A Step Towards Absolute Versions of Metamathematical Results

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IG2 (Philosophical Claim)

No consistent theory (that contains a certain amount of arithmetic) can prove its own consistency. (Shapiro, 2000, p. 167)

G2 (Gödel's Second Theorem)

Let $\mathcal{L}$ contain the language of PA and let $T \supseteq \mathrm{PA}$ be a consistent r.e. $\mathcal{L}$-theory. We have $T \nvdash \neg\mathrm{Pr}(\ulcorner \bot \urcorner)$, for every $\mathcal{L}$-formula $\mathrm{Pr}(x)$ satisfying Löb's derivability conditions for T.

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Question

How can we infer IG2 from G2?

"[T]hough [IG2] does not itself constitute a philosophical application of G2, it is the type of statement upon which such an application must be based. So, to illustrate, while [IG2] does not itself state that G2 refutes Hilbert's Program, it is nonetheless the type of statement to which such an evaluation of Hilbert's Program must need appeal. It is, in a word, what for philosophical purposes we might regard G2 as 'saying'."

(Detlefsen, 2001, p. 40)

"These looser remarks, or sometimes [IG2], are the usual material for philosophical writings concerning the philosophical significance, or consequences, of the Gödel Second Theorem. Since [IG2] isn't the mathematically proved Second Theorem, such writings would be helped by an argument that [IG2] is true."

(Auerbach, 1985, p. 339)

Bridge Principles

P1 Every theory that contains a certain amount of arithmetic is a r.e. $\mathcal{L}$-theory T with $T \supseteq \mathsf{PA}$, where $\mathcal{L}$ contains the language of PA.

P2 An $\mathcal{L}$-theory T proves its own consistency, iff there is a T-provable $\mathcal{L}$-sentence which expresses the consistency of T.

P3 Every $\mathcal{L}$-sentence which expresses the consistency of a theory T is of the form $\neg \mathrm{Pr}(\ulcorner \bot \urcorner)$, where $\mathrm{Pr}(x)$ is some $\mathcal{L}$-formula satisfying Löb's conditions for T.

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The Standard Inference

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The Standard Inference

$$\frac{\text{G2} \qquad \text{P1, P2, P3}}{\text{IG2}}$$

The standard inference is inadequate, since P1 and P3 rely on specific but arbitrary formalisation choices.

P1 relies on a specific notation system for $\mathcal{L}$, e.g., infix notation on strings.

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Alternative notations of $(\forall \mathsf{v}_0(\exists \mathsf{v}_1(\mathsf{S}\mathsf{v}_0 = \mathsf{v}_1)))$:

$$\forall \mathsf{v}_0 \exists \mathsf{v}_1 = \mathsf{S}\mathsf{v}_0\mathsf{v}_1$$

Notation Systems:

- Finite Strings (Tarski, Quine): Polish notation, Peano notation, etc.

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$$(x).(\exists y).x+ = y$$

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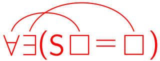

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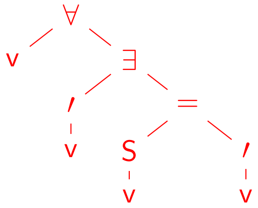

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- Quine-Bourbaki-notation
- Finite Trees (Vermeulen 2000)

Formalisation Dependence: Notation Systems

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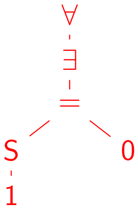

Notation Systems:

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$$\langle \forall, \mathsf{v}, \langle \exists, \langle \prime, \mathsf{v} \rangle, \langle =, \langle \mathsf{S}, \mathsf{v} \rangle, \langle \prime, \mathsf{v} \rangle \rangle \rangle \rangle$$

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- S-Expressions (Kleene 1952), (Feferman 1994)

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$$\forall(v, \exists(\prime(v), =(S(v), e(\prime(v)))))$$

Notation Systems:

- Finite Strings
- Quine-Bourbaki-notation
- Finite Trees
- De Bruijn-index notation
- S-Expressions
- Free Algebras (Hájek and Pudlák 1998), (Béziau 1999)

Claim: Shapiro's application of IG2 requires a robust, i.e., formalisation independent, reading.

> *"[IG2] does indicate trouble for the Hilbert programme. Let PA be a formalization of (ideal) arithmetic, say the classical theory of the natural numbers. The Hilbert programme requires a finitary proof of the consistency of PA. But the second incompleteness theorem is that if PA is in fact consistent, then a straightforward statement of the consistency of PA is not derivable in PA itself, let alone in the finitary portion of PA. The same goes for any other formal system, so long as it contains a certain amount of arithmetic."*
>
> (Shapiro 2000, p. 167)

Formalisation Dependence: Löb's conditions

P3: Every $\mathcal{L}$-sentence which expresses the consistency of a theory T is of the form $\neg\mathrm{Pr}(\ulcorner\bot\urcorner)$, where $\mathrm{Pr}(x)$ satisfies Löb's conditions for T.

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Definition (Löb's Conditions)

Let ι be a notation system for $\mathcal{L}$ with domain D_ι and let α be a numbering of D_ι. A ι-formula $\mathrm{Pr}^\alpha_\iota(x)$ is said to satisfy Löb's conditions *relative to ι and α for T*, in short: $\mathrm{Löb}(T,\iota,\alpha)$, if for all ι-sentences ϕ_ι and ψ_ι:

Löb1 $T_\iota \vdash \phi_\iota$ implies $T_\iota \vdash \mathrm{Pr}^\alpha_\iota(\ulcorner\phi_\iota\urcorner^\alpha)$;

Löb2 $T_\iota \vdash \mathrm{Pr}^\alpha_\iota(\ulcorner\phi_\iota\urcorner^\alpha) \wedge \mathrm{Pr}^\alpha_\iota(\ulcorner\phi_\iota \to \psi_\iota\urcorner^\alpha) \to \mathrm{Pr}^\alpha_\iota(\ulcorner\psi_\iota\urcorner^\alpha)$;

Löb3 $T_\iota \vdash \mathrm{Pr}^\alpha_\iota(\ulcorner\phi_\iota\urcorner^\alpha) \to \mathrm{Pr}^\alpha_\iota(\ulcorner\mathrm{Pr}^\alpha_\iota(\ulcorner\phi_\iota\urcorner^\alpha)\urcorner^\alpha)$.

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According to P3, several standard consistency sentences found in the literature do not express consistency. Example: $\neg\mathsf{Pr}(\ulcorner\bot\urcorner)$, where Pr is Feferman's (1960) standard provability predicate for PA.

Adequate Inferences

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In the context of IG2, replace P1, P2 and P3 with

P1[$\exists\iota$] Every theory that contains a certain amount of arithmetic is a r.e. $\mathcal{L}$-theory T_ι with $T_\iota \supseteq \mathsf{PA}_\iota$, where $\mathcal{L}$ contains the language of PA and ι is some admissible notation system for $\mathcal{L}$.

P2[ι] An $\mathcal{L}$-theory T_ι proves its own consistency, iff there is a T_ι-provable ι-sentence which expresses the consistency of T_ι.

P3[$\exists\alpha\colon D_\iota$] Every ι-sentence which expresses the consistency of a theory T_ι is of the form $\neg\mathrm{Pr}_\iota^\alpha(\ulcorner\bot_\iota\urcorner^\alpha)$, where α is some admissible numbering of the domain D_ι of ι and $\mathrm{Pr}_\iota^\alpha(x)$ is some ι-formula satisfying Löb(T, ι, α).

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Invariance of Gödel's Second Theorem

Problem

The following inference is not valid:

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To fix this, an *invariant* version of G2 is required:

Invariance of G2 - Naïve Approach

Let $\mathcal{L}$ contain the language of PA and let $T \supseteq \text{PA}$ be a consistent r.e. $\mathcal{L}$-theory. We have $T_\iota \nvdash \neg\text{Pr}_\iota^\alpha(\ulcorner \bot_\iota \urcorner^\alpha)$, for every notation system $\langle \mathbf{D}, \iota \rangle$ for $\mathcal{L}$, for every numbering α of $\mathbf{D}$ and for every $\mathcal{L}$-formula $\text{Pr}_\iota^\alpha(x)$ satisfying $\text{Löb}(T, \iota, \alpha)$.

Invariance of Tarski's Theorem

IT (Philosophical Claim)

The collection of all arithmetical truths is not arithmetically definable.

T (Tarski's Theorem)

Let $\mathcal{L}$ contain the language $\mathcal{L}_0$ of PA and let $\mathcal{N}$ be an $\mathcal{L}$-structure which interprets $\mathcal{L}_0$ as usual. The set $\{\phi \in \mathrm{Sent} \mid \mathcal{N} \models \phi\}$ is not definable in $\mathcal{N}$.

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An adequate inference of IT requires the *invariance* of Tarski's Theorem:

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Prima facie, any injective function qualifies as a numbering.

Deviant Numberings

Fix a standard notation system, such as infix notation.

Counterexamples (to Naïve Approach)

There exists a numbering α and a formula $\mathsf{Pr}^{\alpha}(x)$ satisfying $\text{Löb}(T, \alpha)$ such that $T \vdash \neg\mathsf{Pr}^{\alpha}(\ulcorner \bot \urcorner^{\alpha})$.
There exists a numbering β and an $\mathcal{L}_0$-formula $\mathsf{Tr}^{\beta}(x)$ such that $\mathsf{Tr}^{\beta}(x)$ defines $\{\phi \in \mathrm{Sent} \mid \mathcal{N} \models \phi\}$ relative to β.

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- α and β can be chosen to be monotonic;
- there is a numbering α such that $\dot{\mathsf{S}}$, $\dot{+}$, $\dot{\times}$, $\dot{\prime}$, $\dot{=}$, $\dot{\wedge}$, $\dot{\forall}$ are recursive relative to α and $\{\alpha(\phi) \mid T \vdash \phi\}$ is $\Delta_0^0(\exp)$-binumerable;
- there is a numbering β such that $\dot{\mathsf{S}}$, $\dot{+}$, $\dot{\times}$, $\dot{\prime}$, $\dot{=}$, $\dot{\neg}$, $\dot{\wedge}$, $\dot{\rightarrow}$ are recursive relative to β and $\{\beta(\phi) \mid \mathcal{N} \models \phi\}$ is $\Delta_0^0(\exp)$-binumerable.

Definition

For instance, $\dot{\forall}$ is defined via $(x, \phi) \mapsto \forall x\phi$, for expressions x, ϕ. We call a syntactic function f recursive relative to α, if $\alpha \circ f \circ \alpha^{-1}$ is recursive.

Deviant Notation Systems

Deviant results can independently occur on the level of notation systems:

Counterexamples (cont.)

There exists a notation system ι, a standard numbering γ of D_ι and a formula $\mathsf{Pr}_\iota^\gamma(x)$ satisfying $\mathsf{Löb}(T, \iota, \gamma)$ such that $T \vdash \neg\mathsf{Pr}_\iota^\gamma(\ulcorner \bot_\iota \urcorner^\gamma)$.

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Simple example: Let $\mathcal{A}$ be an alphabet for $\mathcal{L}$ and let $\mathcal{A}$ be a copy of $\mathcal{A}$ containing red symbols. Define a notation system ι over $\mathcal{A}^* \cup \mathcal{A}^*$ such that $\phi_\iota \in \mathcal{A}^*$ iff $T \vdash \phi_\iota$. Set $\mathrm{Pr}^\gamma_\iota(x) := \mathrm{Red}(x)$, where $\mathrm{Red}(x)$ is a Δ^0_0-binumeration of the (γ-codes of) red strings.

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The Culprit. The Diagonal Lemma fails for deviant formalisation choices:

- there is no λ such that $T \vdash \lambda \leftrightarrow \neg\mathrm{Pr}^\alpha(\ulcorner \lambda \urcorner^\alpha)$;
- there is no λ_ι such that $T \vdash \lambda_\iota \leftrightarrow \neg\mathrm{Pr}^\gamma_\iota(\ulcorner \lambda_\iota \urcorner^\gamma)$;
- there is no λ such that $\mathcal{N} \models \lambda \leftrightarrow \neg\mathrm{Tr}^\beta(\ulcorner \lambda \urcorner^\beta)$.

Admissibility

These counterexamples do not refute IG2 and IT, since α, β and ι are *inadmissible* formalisation choices.

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Invariance of G2 - Refined Approach

Let $\mathcal{L}$ contain the language of PA and let $T \supseteq$ PA be a consistent r.e. $\mathcal{L}$-theory. We have $T_\iota \nvdash \neg \mathsf{Pr}^\alpha_\iota(\ulcorner \bot_\iota \urcorner^\alpha)$, for every admissible notation system $\langle \mathbf{D}, \iota \rangle$ for $\mathcal{L}$, for every admissible numbering α of $\mathbf{D}$ and for every $\mathcal{L}$-formula $\mathsf{Pr}^\alpha_\iota(x)$ satisfying Löb(T, ι, α).

Invariance of T - Refined Approach

Let $\mathcal{L}$ contain the language $\mathcal{L}_0$ of PA and let $\mathcal{N}$ be an $\mathcal{L}$-structure which interprets $\mathcal{L}_0$ as usual. The set $\{\phi_\iota \in \mathrm{Sent}_\iota \mid \mathcal{N} \models \phi_\iota\}$ is not definable in $\mathcal{N}$ relative to α, for every admissible notation system ι for $\mathcal{L}$ and admissible numbering α of D_ι.

Admissibility is analysed relative to the metamathematical context, the given theory and the given interpretation.

$\mathcal{C}$-Numberings

Let $\mathbb{REC}$ denote the set of (total) recursive functions.
Let $\mathbb{DEF}(\mathcal{N})$ denote the set of numerical functions which are $\mathcal{N}$-definable.
Let $\mathcal{C} = \mathbb{REC}$ or $\mathcal{C} = \mathbb{DEF}(\mathcal{N})$. We say that a numerical relation is in $\mathcal{C}$, if its characteristic function is in $\mathcal{C}$.

Definition

Let $\mathbf{D}$ be an Ω-algebra and let $\alpha\colon |\mathbf{D}| \to \omega$ be injective. We call α a *$\mathcal{C}$-numbering of* $\mathbf{D}$, if

1. $\alpha(|\mathbf{D}|) \in \mathcal{C}$;
2. for each k-ary $\sigma \in \Omega$ there exists a function $\sigma_\alpha \in \mathcal{C}$ ("the tracking function of $\sigma_{\mathbf{D}}$") such that the diagram commutes:

$$\begin{array}{ccc} |\mathbf{D}|^k & \xrightarrow{\sigma_{\mathbf{D}}} & |\mathbf{D}| \\ \downarrow{\scriptstyle \alpha^k} & & \downarrow{\scriptstyle \alpha} \\ \alpha(|\mathbf{D}|)^k & \overset{}{\underset{\sigma_\alpha}{\dashrightarrow}} & \alpha(|\mathbf{D}|) \end{array}$$

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Example ($\mathbf{D} := (\mathcal{A}^*, *)$, $\mathcal{C} := \mathbb{REC}$)

Let $(\mathcal{A}^*, *)$ be the semi-group of $\mathcal{A}$-strings together with concatenation.
A numbering α of $\mathcal{A}^*$ is a $\mathbb{REC}$-numbering of $(\mathcal{A}^*, *)$, iff

1. $\alpha(\mathcal{A}^*) \in \mathbb{REC}$; i.e., $\alpha(\mathcal{A}^*)$ is decidable;
2. for the binary "concatenation" function symbol $\sigma \in \Omega$ there exists a recursive function σ_α such that the diagram commutes:

$$\begin{array}{ccc} \mathcal{A}^* \times \mathcal{A}^* & \xrightarrow{\quad * \quad} & \mathcal{A}^* \\ \Big\downarrow{\scriptstyle \alpha\times\alpha} & & \Big\downarrow{\scriptstyle \alpha} \\ \alpha(\mathcal{A}^*) \times \alpha(\mathcal{A}^*) & \overset{}{\underset{*_\alpha}{\dashrightarrow}} & \alpha(\mathcal{A}^*) \end{array}$$

Admissible Numberings are $\mathcal{C}$-Numberings

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Similarly, in the context of IT and an interpretation $\mathcal{N}$, every admissible numbering is a $\mathbb{DEF}(\mathcal{N})$-numbering.

$\mathcal{C}$-Numberings are $\mathcal{C}$-Equivalent

Definition

Let α and β be numberings of a set A. We say that α is *$\mathcal{C}$-reducible to β* if there exists a function $f \in \mathcal{C}$ such that $\alpha^{-1}(n) = \beta^{-1} \circ f(n)$ for all $n \in \alpha(A)$. We say that α and β are *$\mathcal{C}$-equivalent*, in symbols $\alpha \sim_{\mathcal{C}} \beta$, if α is $\mathcal{C}$-reducible to β and β is $\mathcal{C}$-reducible to α.

Reducibility is a pre-order on the set of numberings of A. Hence, $\sim_{\mathcal{C}}$ is an equivalence relation.

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Theorem (Malcev)

For any $\mathcal{C}$-numberings α and β of a (many-sorted) finitely generated algebra, we have $\alpha \sim_{\mathcal{C}} \beta$.

Invariance of Tarski's Theorem

Corollary

Let $\mathbf{D}$ *be a finitely generated algebra. The* $\mathcal{N}$*-definable subsets of* $|\mathbf{D}|$ *are invariant with regard to* $\mathcal{C}$*-numberings. The decidable and recursive enumerable subsets of* $|\mathbf{D}|$ *are invariant with regard to* $\mathbb{REC}$*-numberings.*

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Fix a standard notation system for $\mathcal{L}$: infix notations on strings in $\mathcal{A}^*$.

Corollary (Invariance of T regarding Numberings)

Let $\mathcal{L}$ *contain* $\mathcal{L}_0$ *and let* $\mathcal{N}$ *be an* $\mathcal{L}$*-structure which interprets* $\mathcal{L}_0$ *as usual. The set* $\{\phi \in \mathrm{Sent} \mid \mathcal{N} \models \phi\}$ *is not definable in* $\mathcal{N}$ *relative to* α*, for every* $\mathbb{DEF}(\mathcal{N})$*-numbering* α *of* $(\mathcal{A}^*, *)$*.*

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Proof.

Let γ be a standard numbering. Then $\{\gamma(\phi) \mid \mathcal{N} \models \phi\}$ is not $\mathcal{N}$-definable. Hence, by the corollary above, $\{\alpha(\phi) \mid \mathcal{N} \models \phi\}$ is not $\mathcal{N}$-definable. □

Lemma

Let α and β be numberings of a set A with $\alpha \sim_{\mathbb{REC}} \beta$ and let $T \supseteq \mathsf{R}$. Then there exists a binumeration $f(x,y)$ of $\beta \circ \alpha^{-1}$ in T such that for each $\phi(x) \in Fml_{\mathcal{L}}$ there is $\psi(x) \in Fml_{\mathcal{L}}$, such that for all $n, m \in \omega$:

$$T \vdash f(\overline{n}, \overline{m}) \to (\phi(\overline{n}) \leftrightarrow \psi(\overline{m})).$$

Moreover, if T is Σ^0_2-sound and $\phi(x)$ is a Σ^0_1-numeration of $\alpha(B)$ in T, with $B \subseteq A$, then $\psi(x)$ numerates $\beta(B)$ in T.

Formalising $\mathbb{REC}$-Equivalence

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Corollary (Invariance of Diagonal Lemma regarding Numberings)

Let α be a $\mathbb{REC}$-numbering and let $\phi(x) \in Fml_{\mathcal{L}}$. Then there exists a sentence λ such that $\mathsf{R} \vdash \phi(\ulcorner \lambda \urcorner^{\alpha}) \leftrightarrow \lambda$.

An "Arithmetisation-Free" Proof of the Diagonal Lemma

Diagonal Lemma

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Define $\delta\colon \mathcal{A}^* \to \omega$ by setting

$$\delta(\chi) = \begin{cases} 2n+1 & \text{if } n = \min\{k \mid \chi \equiv \psi_k(\overline{2k+1})\} \\ 2\gamma(\chi) & \text{if there is no } k \text{ s.t. } \chi \equiv \psi_k(\overline{2k+1}) \end{cases}$$

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1. δ is a $\mathbb{REC}$-numbering;
2. For each $\psi(x) \in Fml_{\mathcal{L}}$ there exists $m \in \omega$ with $\delta(\psi(\overline{m})) = m$.

Proof: Pick $n \in \omega$ such that $\psi \equiv \psi_n$, and then compute minimal k such that $\psi_k(\overline{2k+1}) \equiv \psi_n(\overline{2n+1})$. Output $m := 2k+1$.

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$\Rightarrow$ No arithmetisation of substitution or the numeral function is required!

Invariance regarding Numberings

By the usual "modal reasoning", we get

Theorem (Invariance of G2 regarding Numberings)

Let $\mathcal{L}$ contain the language of PA and let $T \supseteq \mathsf{R}$ be a consistent $\mathcal{L}$-theory. We have $T \nvdash \neg\mathsf{Pr}^{\alpha}(\ulcorner \bot \urcorner^{\alpha})$, for every $\mathbb{REC}$-numbering α of $(\mathcal{A}^*, *)$ and for every $\mathcal{L}$-formula $\mathsf{Pr}^{\alpha}(x)$ satisfying $\mathsf{Löb}(T, \alpha)$.

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Let γ be standard and let $\mathsf{Pr}^{\gamma}(x) :\equiv \bigwedge \mathsf{EA} \to \mathsf{Pr}^{\gamma}_{*}(x)$, where $\mathsf{Pr}^{\gamma}_{*}(x)$ is a standard provability predicate for T.

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Since $\mathsf{Pr}^{\gamma}(x)$ is a numeration of $\gamma(T)$ in T, $\mathsf{Pr}^{\alpha}(x)$ numerates $\alpha(T)$ in T. □

Definition

A *notation system for* $\mathcal{L}$ is a pair $\langle \mathbf{D}, \iota \rangle$, where $\mathbf{D}$ is a many-sorted finitely generated algebra and ι is an implementation of the proto-expressions of $\mathcal{L}$ into $|\mathbf{D}|$.

Example: $\langle (\mathcal{A}^*, *), \iota \rangle$, where ι maps the proto-expressions of $\mathcal{L}$ to infix strings.

Proto-Expressions

Let $\mathtt{S} = \{\mathtt{var}, \mathtt{ter}, \mathtt{fml}\}$ be a set of sorts. For each language $\mathcal{L}$ we define the $\mathtt{S}$-sorted *signature* $\Sigma(\mathcal{L})$ *of constructor symbols for* $\mathcal{L}$ as follows:

1. There is a constant symbol v of sort $\mathtt{var}$ in $\Sigma(\mathcal{L})$;
2. There is a function symbol $\prime$ of type $\mathtt{var} \to \mathtt{var}$ in $\Sigma(\mathcal{L})$;
3. There is a function symbol e of type $\mathtt{var} \to \mathtt{ter}$ in $\Sigma(\mathcal{L})$;
4. For each n-ary function symbol f of $\mathcal{L}$ there is a function symbol f of type $\mathtt{ter}^n \to \mathtt{ter}$ in $\Sigma(\mathcal{L})$;
5. There is a function symbol $=$ of type $\mathtt{ter} \times \mathtt{ter} \to \mathtt{fml}$ in $\Sigma(\mathcal{L})$;
6. For each n-ary propositional connective p of $\mathcal{L}$ there is a function symbol p of type $\mathtt{fml}^n \to \mathtt{fml}$ in $\Sigma(\mathcal{L})$;
7. For each n-ary relation symbol r of $\mathcal{L}$ there is a function symbol r of type $\mathtt{ter}^n \to \mathtt{fml}$ in $\Sigma(\mathcal{L})$;
8. For each quantifier q of $\mathcal{L}$, there is a function symbol q of type $\mathtt{var} \times \mathtt{fml} \to \mathtt{fml}$ in $\Sigma(\mathcal{L})$.

Proto-Expressions

Let $\mathbf{T}_{\Sigma(\mathcal{L})}$ denote the Herbrand algebra of $\Sigma(\mathcal{L})$. We call the elements of $T_{\Sigma(\mathcal{L})}$ the *proto-expressions of* $\mathcal{L}$, or in short: *p-expressions of* $\mathcal{L}$.
In particular, we call $\Sigma(\mathcal{L})$-terms of $T_{\Sigma(\mathcal{L}),\mathtt{var}}$, $T_{\Sigma(\mathcal{L}),\mathtt{ter}}$ and $T_{\Sigma(\mathcal{L}),\mathtt{fml}}$ the *p-variables*, *p-terms* and *p-formulæ* of $\mathcal{L}$ respectively.

Proto-Expressions: ι-equivalence

Let ι be a family $\langle \iota_s \colon T_{\Sigma(\mathcal{L}),s} \to \bigcup_{z \in S} |\mathbf{D}|_z \rangle_{s \in \mathbb{S}}$ of functions.
We define an equivalence relation $\approx_\iota$ on $T_{\Sigma(\mathcal{L})}$, i.e., an equivalence relation on each $T_{\Sigma(\mathcal{L}),s}$, by setting $\phi \approx_\iota \psi$ iff $\iota(\phi) = \iota(\psi)$, for p-expressions ϕ and ψ. We say that ϕ and ψ are *ι-equivalent* iff $\phi \approx_\iota \psi$.

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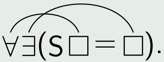
Example

According to Quine-Bourbaki notation, each p-formula of the form

$$\forall(x, \exists(y, = (x, \mathsf{S}(\mathsf{e}(y))))),$$

with p-variables x, y, is implemented as



Hence, ι-equivalence for Quine-Bourbaki notation coincides with α-equivalence, i.e., syntactic identity up to renaming of bounded variables.

Implementations of Proto-Expressions

Definition

Let $\iota = \langle \iota_s \colon T_{\Sigma(\mathcal{L}),s} \to \bigcup_{z \in S} |\mathbf{D}|_z \rangle_{s \in \mathrm{S}}$ and U be an $\mathcal{L}$-theory. We call ι an *implementation of p-expressions of* $\mathcal{L}$ *over* U *into* $\mathbf{D}$, if

1. $\approx_\iota$ is a congruence relation on $\mathbf{T}_{\Sigma(\mathcal{L})}$;
2. If $\phi \approx_\iota \psi$, then $\#\,\mathrm{FV}(\phi) = \#\,\mathrm{FV}(\psi)$ for all p-expressions ϕ, ψ;
3. If $\phi \approx_\iota \psi$ and $\#\,\mathrm{FV}(\phi) = 0$, then $U \vdash \leftrightarrow(\phi, \psi)$, for all p-formulæ ϕ, ψ;
4. If $\phi \approx_\iota \psi$, $\#\,\mathrm{FV}(\phi) = k$ and $s_i \approx_\iota t_i$ for all $i < k$, then
$$\phi(s_0, \ldots, s_{k-1}) \approx_\iota \psi(t_0, \ldots, t_{k-1}),$$
for all p-terms s_i, t_i and p-formulæ ϕ, ψ.

We call $\iota(\phi)$ the *ι-implementation of the p-expression* ϕ *in* $\mathbf{D}$, which is also denoted by ϕ_ι. We also call ϕ_ι a *ι-variable*, *ι-term* and *ι-formula*, if ϕ is a p-variable, p-term and p-formula respectively.

Admissible Notation Systems

To exclude deviant notation systems, we single out *admissible* choices.

Definition

We say that a notation system $\langle \mathbf{D}, \iota \rangle$ for $\mathcal{L}$ is $(\mathcal{C}, U)$-*adequate*, if

1. There is a $\mathcal{C}$-numbering α of $\mathbf{D}$;
2. ι is an implementation over U;
3. $\approx_\iota$ is in $\mathcal{C}$;
4. $\iota(T_{\Sigma(\mathcal{L}),s})$ is in $\mathcal{C}$, for each $s \in \mathtt{S}$;
5. The function $\alpha \circ \tilde{\iota}$ is a $\mathcal{C}$-numbering of $\mathbf{T}_{\Sigma(\mathcal{L})}/\approx_\iota$, where $\tilde{\iota}\colon \mathbf{T}_{\Sigma(\mathcal{L})}/\approx_\iota \to |\mathbf{D}|$ is given by $[\phi]_\iota \mapsto \iota(\phi)$.

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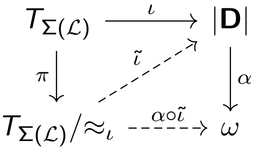
To illustrate, ι and α induce unique injective functions $\tilde{\iota}$ and $\alpha \circ \tilde{\iota}$ such that the following diagram commutes:



Dependency of Fixed Points on the Notation System

For a set T of p-expressions we set $T_\iota := \{\iota(\phi) \mid \phi \in T\}$.
We define $T_\iota \vdash \phi_\iota$ ($\mathcal{N} \models \phi_\iota$), if $T \vdash \phi$ ($\mathcal{N} \models \phi$).
This is well-defined if ι is an implementation of $\mathcal{L}$ over T (over $\mathrm{Th}(\mathcal{N})$).

By the standard Diagonal Lemma, we find for every infix formula $\phi(x)$ an infix sentence λ such that $T \vdash \phi(\ulcorner \lambda \urcorner^\gamma) \leftrightarrow \lambda$, for some standard γ.
Let now $\langle \mathbf{D}, \iota \rangle$ be any notation system for $\mathcal{L}$, where ι is an implementation over T. By definition above, we have $T_\iota \vdash \phi_\iota(\ulcorner \lambda \urcorner^\gamma) \leftrightarrow \lambda_\iota$.
However, this is not the desired Diagonal Lemma for $\langle \mathbf{D}, \iota \rangle$, since the fixed point λ is "used" in ι-notation, while it is "mentioned" in infix notation.

Invariant Diagonalisation

Lemma

Let ι be an implementation of p-expressions of $\mathcal{L}$ such that $\approx_\iota$ is in $\mathcal{C}$ and $\iota(T_{\Sigma(\mathcal{L}),s})$ is in $\mathcal{C}$, for each $s \in \mathtt{S}$. Then there is a self-referential $\mathcal{C}$-numbering δ of $\mathbf{T}_{\Sigma(\mathcal{L})}/\approx_\iota$. That is, for each $\Phi \in T_{\Sigma(\mathcal{L}),\mathtt{fml}}/\approx_\iota$ with exactly one free variable there is a number m such that $\delta(\Phi([\overline{m}])) = m$.

Invariant Diagonalisation

Lemma

Let ι be an implementation of p-expressions of $\mathcal{L}$ such that $\approx_\iota$ is in $\mathcal{C}$ and $\iota(T_{\Sigma(\mathcal{L}),s})$ is in $\mathcal{C}$, for each $s \in \mathtt{S}$. Then there is a self-referential $\mathcal{C}$-numbering δ of $\mathbf{T}_{\Sigma(\mathcal{L})}/{\approx_\iota}$. That is, for each $\Phi \in T_{\Sigma(\mathcal{L}),\mathtt{fml}}/{\approx_\iota}$ with exactly one free variable there is a number m such that $\delta(\Phi([\overline{m}])) = m$.

Invariance of Syntactic Diagonal Lemma

Let $T \supseteq \mathsf{R}$ be an $\mathcal{L}$-theory. Let $\langle \mathbf{D}, \iota \rangle$ be a $(\mathbb{REC}, T)$-adequate notation system for $\mathcal{L}$ and let α be a $\mathbb{REC}$-numbering of $\mathbf{D}$. For every ι-formula ϕ_ι with one free variable there is a ι-sentence λ_ι s.t. $T_\iota \vdash \phi_\iota(\ulcorner \lambda_\iota \urcorner^\alpha) \leftrightarrow \lambda_\iota$.

Invariance of Semantic Diagonal Lemma

Let $\langle \mathbf{D}, \iota \rangle$ be a $(\mathbb{DEF}(\mathcal{N}), \mathrm{Th}(\mathcal{N}))$-adequate notation system for $\mathcal{L}$ and let α be a $\mathbb{DEF}(\mathcal{N})$-numbering of $\mathbf{D}$. For every ι-formula ϕ_ι with one free variable there is a ι-sentence λ_ι such that $\mathcal{N} \models \phi_\iota(\ulcorner \lambda_\iota \urcorner^\alpha) \leftrightarrow \lambda_\iota$.

Theorem (Invariance of G2)

Let $T \supseteq \mathsf{R}$ be a consistent $\mathcal{L}$-theory. We have $T_\iota \nvdash \neg \mathsf{Pr}^\alpha_\iota(\ulcorner \bot_\iota \urcorner^\alpha)$, for every $(\mathbb{REC}, T)$-adequate notation system $\langle \mathbf{D}, \iota \rangle$ for $\mathcal{L}$, for every $\mathbb{REC}$-numbering α of $\mathbf{D}$ and for every $\mathcal{L}$-formula $\mathsf{Pr}^\alpha_\iota(x)$ satisfying $\mathsf{L\ddot{o}b}(T, \iota, \alpha)$.

Invariance of Gödel's Second Theorem

Theorem (Invariance of G2)

Let $T \supseteq \mathsf{R}$ be a consistent $\mathcal{L}$-theory. We have $T_\iota \nvdash \neg\mathsf{Pr}_\iota^\alpha(\ulcorner \bot_\iota \urcorner^\alpha)$, for every $(\mathbb{REC}, T)$-adequate notation system $\langle \mathbf{D}, \iota \rangle$ for $\mathcal{L}$, for every $\mathbb{REC}$-numbering α of $\mathbf{D}$ and for every $\mathcal{L}$-formula $\mathsf{Pr}_\iota^\alpha(x)$ satisfying $\mathsf{L\ddot{o}b}(T, \iota, \alpha)$.

We can now adequately infer the philosophical interpretation IG2:

$$\frac{\text{Invariance of G2} \qquad \mathsf{P1}[\exists\iota],\ \mathsf{P2}[\iota],\ \mathsf{P3}[\exists\alpha\colon D_\iota]}{\mathsf{IG2}}$$

In the context of Gödel's Second Theorem and a theory T,

- every admissible notation system is $(\mathbb{REC}, T)$-adequate;
- every admissible Gödel numbering is a $\mathbb{REC}$-numbering.

Derivation of IG2

(P1[$\exists\iota$]) Every theory that contains a certain amount of arithmetic is a r.e. $\mathcal{L}$-theory T_ι with $T_\iota \supseteq \mathsf{PA}_\iota$, where $\mathcal{L}$ contains the language of PA and ι is some ($\mathbb{REC}$, T)-adequate notation system for $\mathcal{L}$.

(P2[ι]) An $\mathcal{L}$-theory T_ι proves its own consistency, iff there is a T_ι-provable ι-sentence which expresses the consistency of T_ι.

(P3[$\exists\alpha$]) If a ι-sentence expresses the consistency of T_ι, then it is of the form $\neg\mathsf{Pr}_\iota^\alpha(\ulcorner\bot_\iota\urcorner^\alpha)$, where α is an $\mathbb{REC}$-numbering of the domain **D** of ι and $\mathsf{Pr}_\iota^\alpha(x)$ is a ι-formula satisfying Löb(T, ι, α).

(Thm) $T_\iota \nvdash \neg\mathsf{Pr}_\iota^\alpha(\ulcorner\bot_\iota\urcorner^\alpha)$, for every ($\mathbb{REC}$, T)-adequate notation system $\langle \mathbf{D}, \iota\rangle$ for $\mathcal{L}$, for every $\mathbb{REC}$-numbering α of **D** and for every $\mathcal{L}$-formula $\mathsf{Pr}_\iota^\alpha(x)$ satisfying Löb(T, ι, α).

(IG2) No consistent theory (that contains a certain amount of arithmetic) can prove its own consistency.

Invariance of Tarski's Theorem

Theorem (Invariance of T)

For every $(\mathbb{DEF}(\mathcal{N}), \mathsf{Th}(\mathcal{N}))$-adequate notation system $\langle \mathbf{D}, \iota \rangle$ for $\mathcal{L}$ and every $\mathbb{DEF}(\mathcal{N})$-numbering α of **D**, the set $\{\phi_\iota \in \mathrm{Sent}_\iota \mid \mathcal{N} \models \phi_\iota\}$ is not definable in $\mathcal{N}$ relative to α.

Invariance of Tarski's Theorem

Theorem (Invariance of T)

For every $(\mathbb{DEF}(\mathcal{N}), \mathsf{Th}(\mathcal{N}))$-adequate notation system $\langle \mathbf{D}, \iota \rangle$ for $\mathcal{L}$ and every $\mathbb{DEF}(\mathcal{N})$-numbering α of $\mathbf{D}$, the set $\{\phi_\iota \in \mathrm{Sent}_\iota \mid \mathcal{N} \models \phi_\iota\}$ is not definable in $\mathcal{N}$ relative to α.

We can now adequately infer the philosophical interpretation IT:

$$\frac{\text{Invariance of T} \qquad \text{P4}[\exists\iota, \alpha]}{\text{IT}}$$

P4[$\exists\iota, \alpha$] The collection of arithmetical truths is arithmetically definable, if there is a language $\mathcal{L} \supseteq \mathcal{L}_0$ with interpretation $\mathcal{N}$, an admissible notation system $\langle \mathbf{D}, \iota \rangle$ for $\mathcal{L}$ and an admissible numbering α of $\mathbf{D}$ such that $\{\phi_\iota \in \mathrm{Sent}_\iota \mid \mathcal{N} \models \phi_\iota\}$ is definable in $\mathcal{N}$ relative to α.

Invariance of Tarski's Theorem

Theorem (Invariance of T)

For every $(\mathbb{DEF}(\mathcal{N}), \mathsf{Th}(\mathcal{N}))$-adequate notation system $\langle \mathbf{D}, \iota \rangle$ for $\mathcal{L}$ and every $\mathbb{DEF}(\mathcal{N})$-numbering α of $\mathbf{D}$, the set $\{\phi_\iota \in \mathrm{Sent}_\iota \mid \mathcal{N} \models \phi_\iota\}$ is not definable in $\mathcal{N}$ relative to α.

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In the context of Tarski's Theorem and an interpretation $\mathcal{N}$,

- every admissible notation system is $(\mathbb{DEF}(\mathcal{N}), \mathsf{Th}(\mathcal{N}))$-adequate;
- every admissible numbering is a $\mathbb{DEF}(\mathcal{N})$-numbering.

Conclusion

- The justification of prevalent philosophical interpretations of metamathematical theorems require invariance results.
- There are deviant formalisation choices which yield "counterexamples" to Gödel and Tarski's theorems.
- Invariance claims need to be restricted to admissible formalisation choices. Admissibility is highly context-sensitive.
- We can establish the invariance of Gödel's Second Theorem and Tarski's Theorem regarding admissible notation systems and numberings.

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