When Ackermann meets Goodstein

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International Workshop on Gödel's Incompleteness Theorems
Wuhan University

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1982. Kirby and Paris show that Goodstein’s principle is independent from Peano arithmetic

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3. A base change operation $m \mapsto \langle \ell / k \rangle m$

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Base-k Ackermannian terms: Built from 0, $x + y$, $A_x(k, y)$.

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2. Then, choose b maximal such that $A_a b \leq m$.
3. Write $m = A_a b + c$. This is the (simple) Ackermannian representation for m.

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3. What is the **maximal** proof-theoretic strength of termination of an Ackermannian Goodstein process?

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ATR_0: ACA_0 plus "for every set X and ordinal α, the α^{th} Turing jump of X exists."

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  Note: This requires fundamental sequences with the **Bachmann property**, which we have.

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Thus, it is tempting to set $o_k(A_a(k, b) + c) = \varphi_{o_k(a)} o_k(b) + o_k(c)$.

Ackermann vs. Veblen

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Thus, it is tempting to set $o_k(A_a(k, b) + c) = \varphi_{o_k(a)} o_k(b) + o_k(c)$.

But we can get away with much less!

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Goodstein sequences for ACA_0

Recall: The Goodstein sequence starting at m is defined by

- $G_0 m = m$
- $G_{i+1} m = \langle i+3/i+2 \rangle G_i m - 1$, if $G_i m > 0$
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Proof idea. The mapping o_k satisfies the required properties for proving these facts.

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Fix k, set $A_a b = A_a(k, b)$.

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4. Finally, $m \equiv_k A_{a_n} b_n + c$, where $m_n := A_{a_n} b_n$ is such that $A_0 m_n > m$.

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The Ackermannian Goodstein process with sandwiching normal forms always terminates, but this fact is not provable in ATR_0.

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Theorem (Maximality of base change)

Given any base-k Ackermannian term τ, $\ell \geq k$, and $m = |\tau|$, we have that

$$\langle \ell/k \rangle \tau \leq \langle \ell/k \rangle m$$

(where $\langle \ell/k \rangle x$ denotes the base-change for the sandwiching normal forms).

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Let $(m_i)_{i<\alpha}$ be a Goodstein walk. We prove by induction that $m_i \leq G_i m$.

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Define $\mathbb{A}_k(\omega\alpha + b) := A_\alpha b$

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- Even within a fixed notation system, the proof-theoretic strength of termination may vary wildly depending on the precise setup of the Goodstein process.
- However, this proof-theoretic strength has an upper bound given by using 'optimal' normal forms.
- **Question.** Do we still obtain termination for Goodstein walks:
  - If we add multiplication to Ackermannian notation?

Concluding remarks

- Goodstein-like processes can readily be generated by choosing different functions with which to represent natural numbers.
- Even within a fixed notation system, the proof-theoretic strength of termination may vary wildly depending on the precise setup of the Goodstein process.
- However, this proof-theoretic strength has an upper bound given by using 'optimal' normal forms.
- **Question.** Do we still obtain termination for Goodstein walks:
    - If we add multiplication to Ackermannian notation?
    - If we replace the Ackermann function by term exponentiation of the form σ^τ?

Thank you!