Tight Theories

Ali Enayat

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Our story begins with ...

Albert Visser's paper **Categories of theories and interpretations**, *Logic in Tehran*, Lecture Notes in Logic, vol. 26, Association for Symbolic Logic, 2006.

And was continued in ...

My paper **Variations on a Visserian Theme**, in *Liber Amicorum Alberti, a Tribute to Albert Visser*, edited by J. van Eijk, R. Iemhoff, and J. Joosten, College Publications, London, 2016.

The paper **Bi-interpretations in weak set theories**, by Joel David Hamkins and Alfredo Roque Freire, in the *Journal of Symbolic Logic*, 2020 (published online).

There are also connections with ...

Internal Categoricity in Arithmetic and Set Theory, by Jouko Väänänen and Tong Wang, *Notre Dame Journal of Formal Logic*, 2015.

An extension of a theorem of Zermelo, by Jouko Väänänen, *Bulletin of Symbolic Logic*, 2019.

Visser's Theorem

Theorem. (Visser) *Suppose U and V are deductively closed extensions of* PA *(in the same language). If U is a retract of V, then $V \quad U$:*

In particular:

If U and V are bi-interpretable, then $U = V$:

Basics (1)

Suppose U and V are first order theories, and for the sake of notational simplicity, let us assume that U and V are theories that *support a definable pairing function* and *are formulated in relational languages*. We use L_U and L_V to respectively designate the languages of U and V.

(a) An interpretation I of U in V, written:

$$I : U \to V$$

is given by a translation τ of each L_U-formula φ into an L_V-formula φ^{τ} with the requirement that $V \vdash \varphi^{\tau}$ for each $\varphi \in U$, where τ is determined by an L_V-formula $\delta(x)$ (referred to as a *domain formula*), and a mapping $P \mapsto_{\tau} A_P$ that translates each n-ary L_U-predicate P into some n-ary L_V-formula A_P. The translation is then lifted to the full first order language in the obvious way by making it commute with propositional connectives, and subject to:

$$(\forall x\, \varphi)^{\tau} = \forall x(\delta(x) \to \varphi^{\tau}) \text{ and } (\exists x\, \varphi)^{\tau} = \exists x(\delta(x) \wedge \varphi^{\tau}).$$

Basics (2)

U is *interpretable* in V, written $U \trianglelefteq V$, iff there is an interpretation $I : U \to V$. U and V are *mutually interpretable* when $U \trianglelefteq V$ and $V \trianglelefteq U$.

$$\mathrm{ACF}_0 \triangleleft \mathrm{RCF} \triangleleft \mathrm{PA} \triangleleft \mathrm{ZF}.$$

Each interpretation $I : U \to V$ gives rise to an *internal* model construction *that* uniformly *builds a model* $\mathcal{M}^I \models U$ *for any* $\mathcal{M} \models V$.

U is a *retract* of V iff there are interpretations I and J with $I : U \to V$ and $J : V \to U$, and a binary U-formula F such that the following holds for every $\mathcal{M} \models U$:

$$F^{\mathcal{M}} : \mathcal{M} \xrightarrow{\cong} \mathcal{M} := \mathcal{M}^{J \circ I}.$$

Basics (3)

U and V are ***bi-interpretable*** (written as $U = V$) iff there are interpretations I and J that witness that U is a retract of V, and additionally, there is a V-formula G, such that for all $M \models U$ and $N \models V$, we have:

$$F^M : M \xrightarrow{\cong} M^* := (M^J)^I \text{ and}$$
$$G^N : N \xrightarrow{\cong} N^* := (N^I)^J.$$

Bi-interpretability is much stronger than mutual interpretability, i.e.,

$$(U \trianglelefteq V) \wedge (V \trianglelefteq U) \not\Rightarrow U = V.$$

For example, ZF and ZFC are mutually interpretable, but they are not bi-interpretable.

Another example: PA and PA $+ \neg\mathrm{Con(PA)}$.

Basics (4)

Theorem. (Folklore).

(1) PA E ACA_0; *but* ACA_0 5 PA*:*

(2) ZF E GB; *but* GB 5 ZF*:*

Let ZF be the result of deleting the powerset axiom from the axioms of ZF, where ZF is formulated using the schemes of collection and separation (instead of the replacement scheme).

Theorem. (Mostowski 1950s).

(1) ZF $+$ *8x*(*jxj* $@_0$) $=$ $Z_2 + \Pi^1_1$-AC.

(2) ZF $+$ *9* [(inaccess( ) ^ *8x* (*jxj*)] $=$ KM$+\Pi^1_1$-AC.

Basics (5)

Let $\mathrm{ZF}_{\mathrm{fin}}$ be the result of replacing the axiom of infinity in ZF by its negation.

Theorem. (Ackermann 1940, Mycielski 1964, Kaye-Wong 2007)

$$\mathrm{PA} = \mathrm{ZF}_{\mathrm{fin}} + \mathrm{TC}:$$

Theorem. (E-Schmerl-Visser 2011) $\mathrm{ZF}_{\mathrm{fin}}$ PA:

Remark. PA is a retract of $\mathrm{ZF}_{\mathrm{fin}}$.

Solidity, Neatness, and Tightness

T is *solid* iff the following property ( ) holds for all models M, M ; and N of T:

( ) If M $\mathrm{D}_{\mathrm{par}}$ N $\mathrm{D}_{\mathrm{par}}$ M and there is an M-definable isomorphism i_0 : M ! M , then there is an M-definable isomorphism i : M ! N.

T is *neat* iff for any two deductively closed extensions U and V of T (both of which are formulated in the language of T), if U is a retract of V, then V U.

T is *tight* iff for any two deductively closed extensions U and V of T (both of which are formulated in the language of T), if U and V are bi-interpretable, then $U = V$:

solidity) neatness) tightness.

Solidity, neatness, and tightness are all preserved under bi-interpretations.

Solidity of PA (1)

Theorem. (Visser 2006) PA *is solid.*

Proof Outline. Suppose M, M ; and N are models of PA such that:

$$M \;\; \mathsf{D}_{\mathrm{par}} \; N \;\; \mathsf{D}_{\mathrm{par}} \;\; M \quad \text{, and}$$

there is an M-definable isomorphism $i_0 : M \;\; ! \;\; M$:
PA has the key feature that if M j= PA and N j= PA then as soon as $N \;\; \mathsf{E}_{\mathrm{par}} \;\; M$ there is an M-definable initial embedding $j : M \;\; ! \;\; N$.

Solidity of PA (2)

Hence there is an M-definable initial embedding $j_0 : M \ ! \ N$ and an N-definable initial embedding $j_1 : N \ ! \ M$.

Both j_0 and j_1 are surjective.

Suppose not; then $j(M)$ is a proper initial segment of M , where j is the M-definable embedding $j : M \ ! \ M$ given by $j := j_1 \ j_0$:

But then $i_0 \, ^1(j(M))$ is a proper M-definable initial segment of M with no last element.

Contradiction! **QED**

Corollary. $\mathrm{ZF}_{\mathrm{fin}} + \mathrm{TC}$ *is solid.*

Solidity of other theories

Theorem. (E 2016) *The following theories are solid*:

1. Z_2 (*second order arithmetic*).
2. ZF (*Zermelo-Fraenkel set theory).*
3. KM (*Kelley-Morse theory of classes*).
4. Higher order analogues of Z_2 and KM (third order arithmetic, fourth order arithmetic, ..., third order theory of classes, fourth order theory of classes, ...).

Theorem. (E) *Other examples of solid theories include*:

1. $\mathrm{ZF} \setminus \{\mathrm{Infinity}\} + \mathrm{TC}$.
2. "Tarski Arithmetic", i.e., the extension of PA with an axiom stating that T (a new predicate added to the language of arithmetic) satisfies Tarski's inductive conditions of a truth predicate, together with all instances of induction in the extended language $L_{\mathrm{PA}} + T$.

Solidity of Second Order Arithmetic

The first stage of proof of the solidity of Z_2 proceeds as in the proof of solidity of PA and yields

There is an $(\mathcal{M}, \mathcal{A})$-definable *isomorphism* $k_0 : \mathcal{N} \longrightarrow \mathcal{M}$, and

There is an $(\mathcal{N}, \mathcal{B})$-definable *isomorphism* $k_1 : \mathcal{M} \longrightarrow \mathcal{N}$.

The last stage of the proof establishes the surjectivity of $\hat{k}_0$.

The key idea is to take advantage of the embedding: $\hat{k} := \hat{k}_0 \circ \hat{k}_1 \circ \hat{\imath}_0$.

And then to show that $\hat{k}$ is the *identity embedding*, thus establishing the desired surjectivity of $\hat{k}_0$.

Solidity of ZF and KM

Theorem. ZF *is solid.*

Proof ingredients: transitive collapses of set-like well-founded extensional relations, the V_α-hierarchy of sets, Tarski's undefinability of truth, and perseverance!

Theorem. KM *is solid.*

Proof ingredients: Those used in the proofs of solidity of ZF, along with a key idea in the solidity proof of Z_2.

Let ZF be the formulation of ZF using the schemes of separation and collection (instead of replacement).

Corollary. ZF + *9* [(inaccess( ) ^ *8x* (*jxj*)] is solid (recall that this theory is bi-interpretable with $\mathrm{KM} + \Pi^1_1$-AC).

Tightness and Internal Categoricity

The tightness of a theory T can be viewed as an internal completeness of T: T cannot describe a "world" that is bi-interpretable, but not isomorphic with the world described by T.

The tightness of T is closely related to the "internal categoricity" of T, a notion studied by Väänänen and T. Wang, who proved the internal categorcity of PA and ZFC.

The proofs of internal categorcity of PA and ZFC are similar to the proofs of tightness of PA and ZF, but they are not quite the same. In particular, the proof of solidity of ZF involves an appeal to Tarski's Undefinability of Truth Theorem that is absent in the proof of internal categoricity of ZF(C).

As demonstrated by Hamkins and Freire, the internal categoricity of ZF can be used to show the tightness of ZF.

Conjectures/Questions

1. Does every finitely axiomatized sequential theory fail to tight? More generlly, if a sequential theory is tight, then does it prove the full scheme of induction over its natural numbers?
2. (Special case of the above). Is it true that no proper subtheory of PA is tight; in other words, if the deductive closure of T is a subset of the deductive closure of PA and T is tight, then do T and PA have the same deductive closure?
3. Is ZF n fInfinity$g$ + TC the only proper subtheory of ZF that is tight?
4. Is there a classification theorem for sequential tight theories?

Evidence

1. (E 2016) ZF without Extensionality and ZF without Foundation are not tight.
2. (E 2016) ZF_{fin} is not tight.
3. (E) No finitely axiomatizable subtheory of PA or of ZF is tight. Indeed, for each $n \in \omega$, PA_{Π_n} and ZF_{Π_n} fail to be tight; here T_{Π_n} is the set of Π_n-consequences of T.
4. (E) ACA_0 is not tight; the same goes for GB.
5. (Hamkins-Freire 2020) ZF is not tight.
6. (Hamkins-Freire 2020) Zermelo set theory is not tight.

Proof idea of (3)

For $M \models \mathrm{PA}$, $K_n(M)$ is the submodel of M whose universe consists of elements of M that are definable in M by a Σ_n-formula.

Theorem. (Kirby-Paris, Lessan 1978) *Suppose* $n \in \omega$, $n > 0$, *and* M *is a nonstandard model of* PA, then:

1. $K_n(M) \prec_{\Pi_n} M$, hence $K_n(M) \models \mathrm{Th}_{\Pi_{n+1}}(M)$.
2. $K_n(M) \models \mathrm{PA}_{\Pi_{n+1}} + \mathrm{I}\Sigma_{n-1} + \neg \mathrm{B}\Sigma_n$.

Theorem. (Lessan 1978) *The standard cut* ω *is first order definable in* $K_n(M)$.

Proof idea of (3), continued

The arithmetization of Lessan's above theorem can be verified in PA, and yields:

Theorem. *For each $n \in \omega$ the standard model* $\mathbb{N}$ *of* PA *is bi-interpretable with a nonstandard model of the form $K_n(\mathcal{M})$, where $\mathcal{M}$ is a nonstandard model of* PA.

Thus for each $m, n \in \omega$ we can build elementarily inequivalent nonstandard models $\mathcal{M}_0 \models \mathrm{PA}_{\Pi_m}$, and $\mathcal{M}_1 \models \mathrm{PA}_{\Pi_n}$ such that $\mathcal{M}_0$ and $\mathcal{M}_1$ are bi-interpretable, thus showing the failure of solidity of each PA_{Π_n}.

With a little more work, the above yields the failure of tightness of PA_{Π_n} for each $n \in \omega$.

A similar idea works for models of $\mathrm{ZF} + \mathrm{V} = \mathrm{L}$.

Proof idea of (4)

Theorem. ACA_0 *is not tight*.

1. Consider two extensions of ACA, $\mathsf{ACA}_{\text{Atomic}}$ and $\mathsf{ACA}_{\text{Generic}}$:
2. $\mathsf{ACA}_{\text{Atomic}}$ is ACA plus an axiom that ensures that each class is arithmetical, i.e., definable in the ambient N.
3. $\mathsf{ACA}_{\text{Generic}}$ is ACA plus an axiom that ensures that there is an arithmetically generic class G, and each class is definable in $(\mathsf{N}; G)$.

Lemma 1. $\mathsf{ACA}_{\text{Atomic}}$ *is bi-interpretable with Tarski Arithmetic.*

Lemma 2. $\mathsf{ACA}_{\text{Generic}}$ *is also bi-interpretable with Tarski Arithmetic*.

Theorem. $\mathsf{ACA}_{\text{Atomic}}$ *and* $\mathsf{ACA}_{\text{Generic}}$ *are bi-interpretable, but they have distinct deductive closures.*