Minimal Logics for Incompleteness

A personal and biased journey through the landscape of incompleteness

Joost J. Joosten

Universitat de Barcelona

International Workshop on Gödel's incompleteness theorems, e-Wuhan
August 20, 2021

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in Madeira

Outline of this talk

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- G2: For any consistent axiomatisable theory T of some minimal strength,
  - T does not prove the arithmetical sentence $\mathrm{Con}(T)$ which, in a sense, naturally expresses the consistency of T;
  - if moreover T is even better behaved than merely consistent (e.g., ω-consistent, sound, Σ_1-sound), then T will also not prove $\neg\mathrm{Con}(T)$

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- As a generalisation of the question of the provability of consistency, H. Friedman asked to settle the truth and provability of all questions formulated using iterations and logical combinations of consistency.
- Observe that provability can be reduced to truth, since e.g. $\mathsf{PA} \vdash A$ iff $\mathbb{N} \models \Box_{\mathsf{PA}} A$

A first application of modal logic

- Friedman's question can be cast in the signature

$$\mathsf{CF} := \bot \mid (\mathsf{CF} \to \mathsf{CF}) \mid \Box\mathsf{CF}$$

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Theorem (Consequence of Friedman-Goldfarb-Harrington Theorem)

The $\Box_{\mathrm{PA}}(\cdot)$ predicate is Σ^0_1-complete. That is, for each c.e. set A, there is an arithmetical formula $\rho_A(x)$ such that

$$A = \{n \in \mathbb{N} \mid \mathbb{N} \models \Box_{\mathrm{PA}}\Big(\ulcorner \rho_A(\overline{n}) \urcorner\Big)\}.$$

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- Formalised Löb's Theorem (ignoring GNs):

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- Characterise all provably structural properties in two steps
  - $\mathcal{L}_\Box$ with $\mathrm{Form}_\Box := \bot \mid \mathrm{Prop} \mid \mathrm{Form}_\Box \to \mathrm{Form}_\Box \mid \Box \mathrm{Form}_\Box$
  - Define a denotation of $\mathcal{L}_\Box$ formulas inside the $\mathcal{L}_{\mathsf{PA}}$ formulas

Arithmetical realizations

An arithmetical realization is any function $(\cdot)^\star$ taking:

$$\begin{aligned} \text{formulas in } \mathcal{L}_\Box &\to \text{sentences in } \mathcal{L}_{\mathsf{PA}} \\ \text{propositional variables} &\to \text{arithmetical sentences} \\ \text{boolean connectives} &\to \text{boolean connectives} \\ \Box &\to \Box_{\mathsf{PA}} \end{aligned}$$

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Clearly, for any realization $(\cdot)^\star$ we have for example

$$\mathsf{PA} \vdash \Big(\Box(p \to q) \to (\Box p \to \Box q)\Big)^\star$$

since

$$\mathsf{PA} \vdash \Box_{\mathsf{PA}}(p^\star \to q^\star) \to (\Box_{\mathsf{PA}} p^\star \to \Box_{\mathsf{PA}} q^\star)$$

regardless of $(\cdot)^\star$

The Provability Logic of a Theory

- For a c.e. theory T we define

$$\mathsf{PL}(T) := \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } T \vdash (\varphi)^\star\}$$

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A candidate

- GL is the normal modal logic with axioms
  - All classical logical tautologies in $\mathcal{L}_\Box$ like $\Box p \vee \neg\Box p$, etc.
  - All distributions axioms: $\Box(A \to B) \to (\Box A \to \Box B)$,
  - All Löb axioms: $\Box(\Box A \to A) \to \Box A$.
- and rules
  - Modus Ponens $\dfrac{A \to B \quad A}{B}$,
  - Necessitation $\dfrac{A}{\Box A}$.

Solovay's Theorem

Theorem (Solovay, 1976)

Let $\varphi \in \mathcal{L}_\Box$. *Then:*

$$\mathsf{GL} \vdash \varphi$$
$$\Updownarrow$$
$$\mathsf{PA} \vdash (\varphi)^\star \text{ for any arithmetical realization } (\cdot)^\star$$

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Thus, even though PL(PA) is *prima facie* of complexity Π^0_2, it allows for a decidable description

$$\mathsf{GL} = \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } \mathsf{PA} \vdash (\varphi)^\star\}$$

of complexity PSPACE.

True provability logic

- $\mathsf{PA} \nvdash \Box_{\mathsf{PA}}(\ulcorner 0 = 1 \urcorner) \to 0 = 1$
- $\mathbb{N} \models \Box_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \varphi$ for whatever sentence φ

For a c.e. theory T we define

$$\mathsf{TPL}(T) := \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } \mathbb{N} \models (\varphi)^\star\}$$

A priori, complexity above true arithmetic.
However,

$$\mathsf{TPL}(\mathsf{PA}) = \mathsf{GLS}.$$

Here GLS is axiomatised by all theorems of GL and all reflection axioms $\Box A \to A$ with MP as the only rule.

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Or, equivalently $\mathrm{Con}(U) \to \neg\Big(U \rhd \mathrm{Con}(U)\Big)$.

- Positive formulation of G2: $U \rhd (U + \mathrm{InCon}(U))$.

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Theorem (Shavrukov, Berarducci)

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  - $T_n^\lambda := \bigcup_{\alpha<\lambda} T_n^\alpha$ for $\lambda \leq \Lambda$ a limit;
- allows for the definition of proof theoretic ordinals of T relative to $U \subseteq T$:

$$|T|_n^U := \sup\{\alpha \leq \Lambda \mid U_n^\alpha \subseteq T\}$$

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- Certain reasoning in GLP can be interpreted as reasoning about Turing progressions

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- Worms denote (obviously) elements in a modal logic, but also specific points in a well-defined universal model of GLP^0

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- However, this is not provable in PA (Beklemishev)

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- All applications to ordinal analysis are preserved

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- For various arguments using GLP we do not need the full logic but can work in the restricted signature

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- Decision problem drops from PSPACE for GLP to PTIME for RC
- Moreover, RC allows for more general arithmetical interpretations

Solovay for quantified modal logic?

Let $\mathcal{L}_{\Box,\forall}$ be the language of relational quantified modal logic:

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For a c.e. theory T we define

$$\mathrm{QPL}(T) := \{\varphi \in \mathcal{L}_{\Box,\forall} \mid \text{for any } (\cdot)^\bullet,\ \text{we have } T \vdash (\varphi)^\bullet\}$$

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Example: $\Box\forall x P(x) \to \forall x \Box P(\dot{x})$

Degenerate Quantified Provability Logics

If we define $\mathrm{QL}(T) = \{\varphi \in \mathcal{L}_\forall \mid \text{for any } (\cdot)^\bullet,\ \text{we have } T \vdash (\varphi)^\bullet\}$, then it is not hard to see that CQC = QL(PA).
Proof:

- $\subseteq$ if $\pi \vdash_{\mathrm{CQC}} \varphi$, then also $\pi^\bullet \vdash_{\mathrm{CQC}} \varphi^\bullet$, whence $\pi^\bullet \vdash_{\mathrm{PA}} \varphi^\bullet$
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$$\mathsf{QPL}(\mathsf{PA} + \mathsf{Incon}(\mathsf{PA})) = \mathsf{CQC} + \Box\bot$$

Negative results

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Theorem (Vardanyan, 1986 and McGee, 1985)

$$\{closed\ \varphi \in \mathcal{L}_{\Box,\forall} \mid for\ any\ (\cdot)^\bullet,\ we\ have\ \mathsf{PA} \vdash (\varphi)^\bullet\}$$

is Π^0_2*-complete. Thus it is not recursively axiomatisable.*

Theorem (Artemov, 1985)

TQPL(PA) *is not arithmetical.*

Theorem (Vardanyan, 1985)

TQPL(PA) *is* Π^0_1 *complete in true arithmetic.*

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Vardanyan: intersecting over all possible elementary representations of a theory still yields Π^0_2-completeness

Theorem (Visser, de Jonge, 2006)

$\mathsf{QPL}(T)$ *is* Π^0_2 *complete for any* Σ_1*-sound theory* T *extending* EA.

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No Escape from Vardanyan's Theorem

Planning an escape (jointly with Ana Borges)

Another escape was already known to Artemov and Japaridze: formulas refutable on finite models, in particular, one variable fragment

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Prove arithmetical soundness and completeness for QRC_1:

$$\mathsf{QRC}_1 = \{\varphi \vdash \psi \mid \text{for any } (\cdot)^*,\ \text{we have } \mathsf{PA} \vdash (\varphi \vdash \psi)^*\}$$

QRC_1: Axioms and rules

$$\varphi \vdash \top \qquad \varphi \wedge \psi \vdash \varphi$$

$$\varphi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi$$

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi} \qquad \frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}$$

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t free for x in φ and ψ $\qquad$ c not in φ nor ψ

Some provable and unprovable statements

$$\Diamond \forall x\, \varphi \vdash \forall x\, \Diamond \varphi$$

$$\forall x\, \Diamond \varphi \nvdash \Diamond \forall x\, \varphi$$

$$\frac{\varphi \vdash \psi[x \leftarrow c]}{\varphi \vdash \forall x\, \psi}$$

x not free in φ and c not in φ nor ψ

Arithmetical semantics

The arithmetical realizations $(\cdot)^*$ for $\mathcal{L}_{\Diamond,\forall}$:

$$\begin{aligned} \text{formulas in } \mathcal{L}_{\Diamond,\forall} &\to \text{axiomatisations of c.e. theories in } \mathcal{L}_{\mathrm{PA}} \\ \text{variables } x_i &\to \text{variables } y_i \\ \text{constants } c_i &\to \text{variables } z_i \end{aligned}$$

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(\varphi(x,c) \vdash \psi(x,c))^* &:= \forall\, \theta, y, z\, (\Box_{\psi^*(y,z)}\theta \to \Box_{\varphi^*(y,z)}\theta)
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Theorem (Arithmetical soundness (Ana de Almeida Borges, JjJ))

$$\mathsf{QRC}_1 \subseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \ \textit{we have}$$
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By induction on the QRC_1-proof. Here is the case of $\Diamond\Diamond\varphi \vdash \Diamond\varphi$:

- Pick any $(\cdot)^*$, reason in T, and let θ, y, z be arbitrary
- Assume $\Box_{(\Diamond\varphi)^*}\theta$
- Then $\Box_{\mathrm{PA}}(\mathrm{Con}_{\varphi^*}(\top) \to \theta)$
- By provable Σ_1-completeness, $\Box_{\mathrm{PA}}(\mathrm{Con}_{\mathrm{PA}}(\mathrm{Con}_{\varphi^*}(\top)) \to \mathrm{Con}_{\varphi^*}(\top))$
- Then $\Box_{\mathrm{PA}}(\mathrm{Con}_{\mathrm{PA}}(\mathrm{Con}_{\varphi^*}(\top)) \to \theta)$
- We conclude $\Box_{(\Diamond\Diamond\varphi)^*}\theta$
- Σ_1-collection is needed for $\dfrac{\varphi \vdash \psi}{\varphi \vdash \forall x \psi}$ with $x \notin \varphi$

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$$\mathsf{QRC}_1 \supseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \ \textit{we have } T \vdash (\varphi \vdash \psi)^*\}$$

Where T is a sound r.e. theory extending $\mathsf{I}\Sigma_1$.

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Advanced conjecture: works for HA as well

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- However, IL^0 is still PSPACE complete (Bou, JjJ)

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    - $\mathrm{I}\Sigma_1 \rhd (\mathrm{PRA} + \mathrm{Con}(\mathrm{PRA}))$ (No)

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  - Constant for the sentence axiomatising $I\Sigma_1$ to the closed fragment where $\top$ refers to PRA to obtain interpretability logics PIL and PILS (JjJ)
  - The latter can decide all questions of the form
    - $I\Sigma_1 \rhd (PRA + Con(PRA))$ (No)
    - $(PRA + Con(PRA)) \rhd I\Sigma_1$ (Yes)

Closed fragment with constants

- Sometimes, e.g. when the closed fragment is not strong enough, adding some constants can be more beneficial than resorting to the full propositional language.
- Some biased examples:
  - Constants for ordinals to obtain Turing Schmerl Calculus TSC to *denote* Turing progressions rather than approximating them using GLP or RC (Hermo Reyes, JjJ)
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  - The latter can decide all questions of the form
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    - $(\mathrm{PRA} + \mathrm{Con}(\mathrm{PRA})) \rhd \mathrm{I}\Sigma_1$ (Yes)
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    - etc.

Simplified simplified

Final example: closed fragment of RC

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Here we can even dispense with conjunctions!!!

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Definition (Worm Calculus, WC (Ana de Almeida Borges, JjJ))

Let A, B and C be worms, and n, m be numbers.
The axioms of WC are:

1. $A \vdash_{\mathrm{WC}} \top$;
2. $\langle n \rangle \langle n \rangle A \vdash_{\mathrm{WC}} \langle n \rangle A$ (Transitivity);
3. $\langle n \rangle A \vdash_{\mathrm{WC}} \langle m \rangle A$ for $n > m$ (Monotonicity).

The rules of WC are:

1. If $A \vdash_{\mathrm{WC}} B$ and $B \vdash_{\mathrm{WC}} C$, then $A \vdash_{\mathrm{WC}} C$ (Cut);
2. If $A \vdash_{\mathrm{WC}} B$, then $nA \vdash_{\mathrm{WC}} nB$ (Necessitation);
3. If $A \vdash_{\mathrm{WC}} B$ and $A \vdash_{\mathrm{WC}} nC$, then $A \vdash_{\mathrm{WC}} B\alpha C$, for $B \in \mathbb{W}_{n+1}$.

Thank you

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