Completions of PA and ω-models of KP

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Gödel-Rosser

Theorem (Gödel-Rosser). If A is a computable set of sentences s.t. $PA \cup A$ is consistent, then there is a sentence φ s.t. φ and $\neg\varphi$ are both consistent with $PA \cup A$. Moreover, we can find φ effectively, given an index for A.

Sentence: φ refers to itself, saying "For any proof of me from $PA \cup A$, there is a smaller proof of my negation from these same axioms.

We will return to the Gödel-Rosser sentence, and variants of it.

Subtrees of $2^{<\omega}$ and $\omega^{<\omega}$

For our purposes, a *tree* is a set of finite sequences, closed under initial segment.

1. The tree $2^{<\omega}$ is the set of all finite sequences from the set $2 = \{0, 1\}$. The paths through $2^{<\omega}$ are the elements of *Cantor space* 2^{ω}.

2. The tree $\omega^{<\omega}$ is the set of all finite sequences from the set ω of all natural numbers. The paths through $\omega^{<\omega}$ are the elements of *Baire space* ω^{ω}.

We consider subtrees of these two trees.

Trees of special interest

We focus particularly on the following:

1. a computable tree $\mathcal{T}_{PA} \subseteq 2^{<\omega}$ whose paths represent completions of first order Peano Arithmetic (PA),
2. a computable tree $\mathcal{T}_{KP} \subseteq \omega^{<\omega}$ whose paths represent complete diagrams of ω-models of Kripke-Platek set theory (KP).

Both completions of PA and ω-models of KP have a great deal of self-awareness.

$\mathcal{T}_{PA}$

Proposition. There is a computable tree $\mathcal{T}_{PA} \subseteq 2^{<\omega}$ whose paths represent completions of PA.

Construction: Let $(\varphi_n)_{n\in\omega}$ be a computable list of all sentences in the language of arithmetic. For $\sigma \in 2^s$, $\sigma \in \mathcal{T}_{PA}$ if no $c \leq s$ is a proof of $\perp$ from $PA \cup \{\varphi_n : \sigma(n) = 1\} \cup \{\neg\varphi_n : \sigma(n) = 0\}$.

Representable sets

Definition. Let T be a theory in the language of arithmetic. A set $S \subseteq \omega$ is *representable* with respect to T if there is a formula $\varphi(x)$ s.t.

- if $n \in S$, then $T \vdash \varphi(n)$ and
- if $n \notin S$, then $T \vdash \neg\varphi(n)$.

We write $Rep(T)$ for the family of these sets.

$Rep(PA)$ is the family of computable sets. The same is true for any computably axiomatized extension of PA, or a weak sub-theory. For True Arithmetic (TA), $Rep(TA)$ is the family of all arithmetical sets. Scott characterized the families of sets that can serve as $Rep(T)$ for a completion T of PA.

Scott

Definition. A *Scott set* is a family $S \subseteq P(\omega)$ s.t.

1. If $A, B \in S$, then $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\} \in S$.
2. If $A \in S$ and $B \leq_T A$, then $B \in S$.
3. If $\mathcal{T} \subseteq 2^{<\omega}$ is a tree (coded by a set) in S, then $\mathcal{T}$ has a path in S.

Theorem (Scott, 1962). The families of sets that can serve as $Rep(T)$ for T a completion of PA are exactly the countable Scott sets.

Ingredients of Scott's proof

It is straightforward to show that for each completion T of PA, $Rep(T)$ is a Scott set.

To show that for each countable Scott set S, there is a completion T of PA with $Rep(T) = S$, Scott obtained, for each n, an "independent" formula $\varphi_n(x)$, which can code an arbitrary set, leaving the Π_n part of the theory unchanged. The formula $\varphi_n(x)$ is based on iterates of the Gödel-Rosser sentence, starting with PA and the true $\Pi_n \cup \Sigma_n$ sentences.

Scott and universality

The result below follows from the fact that for each completion T of PA, $Rep(T)$ is a Scott family.

Proposition (Scott). Any path through $\mathcal{T}_{PA}$ computes a path through each computable tree $\mathcal{T} \subseteq 2^{<\omega}$ that has paths. Moreover, there is a uniform effective procedure that yields a path through $\mathcal{T}$, given a path through $\mathcal{T}_{PA}$ and a computable index for $\mathcal{T}$.

Binns-Simpson and non-universality

Binns and Simpson showed the following.

Theorem (Binns-Simpson, 2004). For any computable tree $\mathcal{T}_1 \subseteq \omega^{<\omega}$, with paths, there is a computable tree $\mathcal{T}_2 \subseteq \omega^{<\omega}$, also with paths, s.t. some path through $\mathcal{T}_1$ does not compute any path through $\mathcal{T}_2$.

KP and ω-models

Kripke-Platek set theory is a weak version of set theory, with the usual axioms of extent, pairing, union, and infinity, plus schemata for induction, Δ_0-separation, and Δ_0-collection.

The axiom of infinity guarantees that each model has a set ω.

Definition. An *ω-model* of KP is a model in which the elements of ω are all standard—each has just finitely many predecessors.

In an ω-model of KP, all of the computable, even all of the hyperarithmetical, sets are present.

Ordinals in ω-models of KP

- There is a least ω-model of KP—$L_{\omega_1^{CK}}$. The set of ordinals in $L_{\omega_1^{CK}}$ has order type ω_1^{CK}.
- There are ω-models in which the set of ordinals has the order type of the Harrison ordering.

Recall: A *Harrison ordering* is a computable ordering of type $\omega_1^{CK}(1+\eta)$, with no infinite hyperarithmetical decreasing sequence.

$\mathcal{T}_{KP}$

Proposition. There is a computable tree $\mathcal{T}_{KP} \subseteq \omega^{<\omega}$ whose paths represent complete diagrams of ω-models of KP.

Idea of construction: Each node in $\mathcal{T}_{KP}$ corresponds to a finite set of sentences in the language of set theory with added constants from an infinite computable set C. The node decides the first few sentences, and does some witnessing of sentences that start with either $\exists$ or $\forall$. When we put in a sentence saying $c \in \omega$, we add a sentence saying that $c = n$ for some n. We maintain consistency of the set of quantifier-free sentences.

Tree rank (foundation rank)

Definition. For a tree $\mathcal{T}$ and $\sigma \in \mathcal{T}$,

(1) $rk(\sigma) = 0$ if σ has no successors,

(2) for $\alpha > 0$, $rk(\sigma) = \alpha$ if σ has successors, all of ordinal rank, and α is the first ordinal greater than the ranks of all successors of σ,

(3) $rk(\sigma) = \infty$ if σ does not have ordinal rank.

We define $rk(\mathcal{T})$ to be the rank of the top node $\emptyset$ in$\mathcal{T}$.

Path or rank?

Fact (*ZFC*): For a tree $\mathcal{T} \subseteq \omega^{<\omega}$, $\mathcal{T}$ has a path iff it is unranked.

Proposition (Barwise). If $\mathcal{T}$ is a computable tree with no path, then $rk(\mathcal{T})$ is a computable ordinal.

Computable trees in ω-models of KP

An ω-model of KP calculates computable ordinal ranks just as we do. Suppose $\mathcal{T}$ is a computable tree. If $rk(\mathcal{T}) = \alpha$ in the real world, then $rk(\mathcal{T}) = \alpha$ in ω-models of KP. If $\mathcal{T}$ has no path in the real world, then $\mathcal{T}$ has no path in ω-models of KP.

Fact: The theorem saying that a tree has a path iff it is unranked may fail in ω-models of KP.

Example: Let $\mathcal{T}$ be a computable tree that has paths but no hyperarithmetical path. Then in $L_{\omega_1^{CK}}$, $\mathcal{T}$ is unranked, with no path.

Trees with paths but no hyperarithmetical paths

Here are some computable trees with paths but no hyperarithmetical paths.

1. The tree $\mathcal{T}_{KP}$.

2. The tree $\mathcal{T}_H$ consisting of finite decreasing sequences in a Harrison ordering H.

A special Harrison ordering

For later use, we consider a special Harrison ordering.

Theorem (Goncharov-Harizanov-K-Shore, 2004).

(1) The Turing degrees of the well-ordered parts of Harrison orderings are the same as the degrees of paths through Kleene's O.

(2) There is a path through O that does not compute $\emptyset'$.

Consequence. For a Harrison ordering H in which the well-ordered part W does not compute $\emptyset'$, $\mathcal{T}_H$ has a path f that does not compute $\emptyset'$. Moreover, we may take f extending any finite decreasing sequence in H-W.

Existence of ω_1^{CK}

Theorem. In ω-models of KP, the following are equivalent:

1. Every computable tree is ranked or has a path,
2. ω_1^{CK} exists; i.e., there is a first ordinal not isomorphic to any computable ordering.

Moreover, for an ω-model M of KP, M satisfies (1) and (2) above iff there exists $N \in M$ s.t. N, with the restriction of $\in$, is a transitive model of KP.

Trees with non-standard rank

Proposition. There are ω-models of *KP* in which some computable trees have non-standard ordinal rank.

Proof: Let $\mathcal{T}$ be a computable tree with paths but no hyperarithmetical path. Then $\mathcal{T}$ has nodes of all computable ordinal ranks.

The Barwise-Kreisel Compactness Theorem gives an ω-model *M* of *KP* s.t. some $\sigma \in \mathcal{T}$ has non-standard rank. Then $\mathcal{T}_\sigma = \{\tau : \sigma\tau \in \mathcal{T}\}$ has non-standard rank.

Computing paths

Note. Let $\mathcal{T}$ be a computable tree, and let M be an ω-model of KP in which $\mathcal{T}$ is unranked. Then $D^c(M)$ computes a path through $\mathcal{T}$.

Theorem (Weisshaar, 2019). Let $\mathcal{T}_1, \mathcal{T}_2$ be computable trees, and let M be an ω-model of KP in which $\mathcal{T}_1, \mathcal{T}_2$ have non-standard ranks, where $rk(\mathcal{T}_1) \leq rk(\mathcal{T}_2)$. If f is a path through $\mathcal{T}_1$, then $f \oplus D^c(M)$ computes a path through $\mathcal{T}_2$.

Can we drop $D^c(M)$?

Theorem. There are computable trees $\mathcal{T}_1, \mathcal{T}_2$ s.t. in some ω-model M of KP, $rk(\mathcal{T}_1) < rk(\mathcal{T}_2)$ and some path f through $\mathcal{T}_1$ does not (by itself) compute a path through $\mathcal{T}_2$.

Example: Let $\mathcal{T}_H$ be the tree of decreasing sequences in a Harrison ordering H whose well-ordered part does not compute $\emptyset'$. Take an ω-model M of KP, in which some $\sigma \in \mathcal{T}_H$ and $\tau \in \mathcal{T}_{KP}$ both have non-standard rank and $rk(\sigma) < rk(\tau)$. Let $\mathcal{T}_1$ be the tree below σ in $\mathcal{T}_H$, let $\mathcal{T}_2$ be the tree below τ in $\mathcal{T}_H$, and let f be a path through $\mathcal{T}_1$ that does not compute $\emptyset'$.

Unranked tree

Theorem. There is a computable tree $\mathcal{T}$, s.t. $\mathcal{T}$ has no hyperarithmetical path, and $\mathcal{T}$ is not ranked in any ω-model of KP.

Ingredients of Proof: The proof uses Kleene's Recursion Theorem. It seems to me that Kleene's result must have been inspired by Gödel's self-referential lemma.

The tree $\mathcal{T}$ is constructed so that its paths represent "jump structures" over an ordering of type $\omega + KB(\mathcal{T})$, where $KB(\mathcal{T})$ is the Kleene-Brouwer ordering on $\mathcal{T}$.

Making Binns-Simpson more concrete

By Scott's results, every completion of PA computes a completion of each computably axiomatized extension of PA.

The analogue of this statement fails. For a sentence θ s.t. $KP \cup \{\theta\}$ is ω-consistent, let $\mathcal{T}_{KP+\theta}$ be the sub-tree of $\mathcal{T}_{KP}$ whose paths represent complete diagrams of ω-models of $KP \cup \{\theta\}$.

Theorem. There is a sentence θ s.t. $KP + \theta$ is ω-consistent but some path through $\mathcal{T}_{KP}$ does not compute any path through $\mathcal{T}_{KP+\theta}$.

Is the ordering on ranks determined by the trees?

Theorem. There are computable trees $\mathcal{T}_1, \mathcal{T}_2$ and ω-models M_1, M_2 of KP s.t. $M_1 \models rk(\mathcal{T}_1) < rk(\mathcal{T}_2) < \infty$ and $M_2 \models rk(\mathcal{T}_2) < rk(\mathcal{T}_1) < \infty$. Moreover, there is no ω-model M of KP s.t. $M \models rk(\mathcal{T}_1) = rk(\mathcal{T}_2) < \infty$.

For this, we need the following.

Lemma [self-reference for two variables]. Let $\beta_1(x, y)$ and $\beta_2(x, y)$ be formulas with free variables x, y. Then there are sentences σ_1 and σ_2 s.t. over KP, $\sigma_1 \leftrightarrow \beta_1(\#\sigma_1, \#\sigma_2)$ and $\sigma_2 \leftrightarrow \beta_2(\#\sigma_1, \#\sigma_2)$.

More on tree rank in ω-models of KP

Theorem. There are computable trees $\mathcal{T}_1, \mathcal{T}_2$, each ranked in some ω-model of KP, but s.t. there is no ω-model of KP in which both trees have rank.

Independence and number of paths

We can use the Gödel-Rosser Theorem to prove the following:

Fact. There are $2^{\aleph_0}$ completions of PA.

Analogue of Gödel-Rosser Theorem

Theorem. For a computable set A of axioms, $\mathcal{T}_{KP\cup A}$ is a computable tree whose paths represent complete diagrams of ω-models of $KP \cup A$. Let φ_A be the sentence that refers to itself, saying $rk(T_{KP\cup A+\varphi}) \leq rk(T_{KP\cup A+\neg\varphi})$. If $KP \cup A$ is ω-consistent, then so are $KP \cup A \pm \varphi$.

Corollary. There are $2^{\aleph_0}$ ω-consistent completions of KP.

Number of models

Each completion of PA has $2^{\aleph_0}$ pairwise non-isomorphic models.

For ω-consistent completions of KP, the number of non-isomorphic ω-models varies.

(1) $Th(L_{\omega_1})$ has $2^{\aleph_0}$ non-isomorphic ω-models.

(2) $Th(L_{\omega_1^{CK}})$ has just one ω-model, up to isomorphism.

Non-minimality

Fact. For any completion T_1 of PA, there is another completion T_2 of strictly lower degree.

The analogous statement about complete diagrams of ω-models of KP is also true.

Theorem. Let M be an ω-model of KP. There is another ω-model N of KP s.t. $D^c(N) <_T D^c(M)$.

More non-minimality

Theorem (K, 2001). For any non-standard model M of PA, there is some $N \cong M$ s.t. $D(N) <_T D(M)$.

Analogue?