Inclusions between quantified provability logics

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Outline

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- **Let $\mathcal{L}_A = \{0, S, +, \times, <, =\}$ be the language of first-order arithmetic.**
- **In this talk, T, T_0 and T_1 always denote recursively enumerable $\mathcal{L}_A$-theories extending $\mathbf{I\Sigma_1}$.**
- **Let $\mathsf{Th}(T)$ be the set of all $\mathcal{L}_A$-sentences provable in T.**
- **Let $\mathrm{Pr}_T(x)$ be a natural provability predicate of T.**

Fact

For any formulas φ and ψ,

1. $T \vdash \varphi \Rightarrow \mathbf{I\Sigma_1} \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$
2. $\mathbf{I\Sigma_1} \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$
3. $\mathbf{I\Sigma_1} \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$
4. $\mathbf{I\Sigma_1} \vdash \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$ **(Löb's theorem)**

GL

These properties of $\mathrm{Pr}_T(x)$ can be described using modal logic.

Definition (GL)

The axioms and rules of the modal propositional logic GL are as follows:

A1 **All tautologies**

A2 $\Box(A \to B) \to (\Box A \to \Box B)$

A3 $\Box(\Box A \to A) \to \Box A$

R1 $\dfrac{A \quad A \to B}{B}$ **(Modus ponens)**

R2 $\dfrac{A}{\Box A}$ **(Necessitation)**

Arithmetical interpretations

To connect arithmetic and modal logic precisely, I introduce the notion of arithmetical interpretation.

Definition (arithmetical interpretation)

A mapping f from the set of all propositional variables to the set of $\mathcal{L}_A$-sentences is called an arithmetical interpretation.

Each arithmetical interpretation f is uniquely extended to a mapping f_T from the set of all propositional modal formulas to the set of $\mathcal{L}_A$-sentences inductively as follows:

1. $f_T(\bot)$ **is** $0 = 1$;
2. f_T **commutes with each propositional connective;**
3. $f_T(\Box A)$ **is** $\mathrm{Pr}_T(\ulcorner f_T(A) \urcorner)$.

Propositional provability logic and Solovay's theorem

Definition (propositional provability logic)

$\mathsf{PL}(T) := \{A \mid \forall f\text{: arithmetical interpretation}, T \vdash f_T(A)\}$ is the **propositional provability logic** of T.

Proposition (arithmetical soundness)

For any theory T, $\mathbf{GL} \subseteq \mathsf{PL}(T)$.

Solovay's arithmetical completeness theorem states that the converse inclusion holds for many theories.

Arithmetical completeness theorem (Solovay, 1976)

If T is Σ_1-sound, then $\mathsf{PL}(T) = \mathbf{GL}$.

More on Solovay's theorem

Moreover, Visser listed all the possibilities for $\mathsf{PL}(T)$.

Definition

The sequence $(\mathrm{Con}^n_T)_{n\in\omega}$ of Π_1 sentences is defined as follows:

- Con^0_T **is the sentence** $0 = 0$;
- Con^{n+1}_T **is the sentence** $\mathrm{Con}_{T+\mathrm{Con}^n_T}$.

Theorem (Visser, 1984)

- $\mathsf{PL}(T) = \mathbf{GL} \iff T + \{\mathrm{Con}^n_T \mid n \geq 0\}$ **is consistent;**
- $\mathsf{PL}(T) = \mathbf{GL} + \Box^n \bot \iff n = \min\{k \mid T \vdash \neg\mathrm{Con}^k_T\}$.

$\Box^n\bot$ **is** $\underbrace{\Box\cdots\Box}_{n}\bot$.

From Solovay's and Visser's theorems, we have:

- $\mathsf{PL}(T)$ **is a primitive recursive set.**
- $\mathsf{PL}(T)$ **depends only on the least** n **such that** $T \vdash \neg\mathrm{Con}_T^n$**, and therefore depends very little on the theory** T **itself.**
- **Since** $\mathbf{GL} + \Box^m\bot \subseteq \mathbf{GL} + \Box^n\bot \iff m \geq n$**, for any theories** T_0 **and** T_1**,**

$$\mathsf{PL}(T_0) \subseteq \mathsf{PL}(T_1) \textbf{ or } \mathsf{PL}(T_1) \subseteq \mathsf{PL}(T_0).$$

- **By extending the framework of the argument to predicate logic, the provability logic of T may become dependent on the theory T and have more fine-grained properties regarding the provability predicate $\mathrm{Pr}_T(x)$ of T.**
- **Many works on quantified provability logic were done, especially in the 1980s.**

The language of quantified modal logic

The language of quantified modal logic

- **The language of quantified modal logic is the language of first-order predicate logic without function and constant symbols equipped with the unary modal operators $\Box$ and $\Diamond$.**
- **The languages of quantified modal logic and first-order arithmetic have the same variables.**

Arithmetical interpretation

Definition (arithmetical interpretation)

A mapping f from the set of all atomic formulas of quantified modal logic to the set of $\mathcal{L}_A$-formulas satisfying the following condition is called an arithmetical interpretation: For each atomic formula $P(x_1, \ldots, x_n)$,

- $f(P(x_1, \ldots, x_n))$ **is an $\mathcal{L}_A$-formula $\varphi(x_1, \ldots, x_n)$ with the same free variables;**
- $f(P(y_1, \ldots, y_n))$ **is $\varphi(y_1, \ldots, y_n)$ for any variables $y_1, \ldots, y_n$.**

Each arithmetical interpretation f is uniquely extended to a mapping f_T from the set of all quantified modal formulas to the set of $\mathcal{L}_A$-formulas inductively as follows:

1. $f_T(\bot)$ **is** $0 = 1$;
2. f_T **commutes with each propositional connective and quantifier;**
3. $f_T(\Box A(x_1, \ldots, x_n))$ **is the formula** $\mathrm{Pr}_T(\ulcorner f_T(A(\dot{x}_1, \ldots, \dot{x}_n))\urcorner)$.

Quantified provability logic

Definition (quantified provability logic)

$\mathsf{QPL}(T)$
$:= \{A \mid A$: **sentence and** $\forall f$: **arithmetical interpretation**, $T \vdash f_T(A)\}$
is the quantified provability logic of T**.**

Proposition (arithmetical soundness)

For any theory T, $\mathbf{QGL} \subseteq \mathsf{QPL}(T)$.

- **Does** $\mathsf{QPL}(\mathbf{PA}) \subseteq \mathbf{QGL}$ **hold?** **(Avron, 1984)**
- **Is** $\mathsf{QPL}(\mathbf{PA})$ **r.e.?** **(Boolos, 1979)**

Vardanyan's theorem

Vardanyan gave a negative answer to these questions.

Theorem (Vardanyan, 1985)

QPL(**PA**) is Π^0_2-complete.

Montagna's theorem

$\mathsf{QPL}(T)$ **may heavily depends on the theory** T**.**

Theorem (Montagna, 1984)

If T_1 **is finitely axiomatizable,** $T_1 \nvdash \neg \mathrm{Con}_{T_1}$ **and** $T_0 \vdash \mathrm{Con}_{T_1} \to \mathrm{Con}^2_{T_0}$**, then** $\mathsf{QPL}(T_0) \not\subseteq \mathsf{QPL}(T_1)$**.**

Example

For $0 < i < j$, $\mathsf{QPL}(\mathbf{I\Sigma_i}) \not\subseteq \mathsf{QPL}(\mathbf{I\Sigma_j})$**.**

Notice that $\mathsf{PL}(\mathbf{I\Sigma_i}) = \mathsf{PL}(\mathbf{I\Sigma_j}) = \mathbf{GL}$**.**

Moreover, QPL(T) also depends on Σ_1 formulas defining T.

Definition (Σ_1 definition)

We say a formula $\tau(v)$ is a definition of a theory T if for any natural number n,

$$\mathbb{N} \models \tau(\overline{n}) \iff n \text{ is the Gödel number of some axiom of } T.$$

A Σ_1 formula defining T is called a Σ_1 definition of T.

Let $\tau(v)$ be a Σ_1 definition of T.

- **We can construct a Σ_1 provability predicate $\mathrm{Pr}_\tau(x)$ of T saying that "x is provable in the theory defined by $\tau(v)$".**
- **For each arithmetical interpretation f, the mapping obtained by extending f by using $\mathrm{Pr}_\tau(x)$ is denoted by f_τ.**
  That is, $f_\tau(\Box A(x_1, \ldots, x_n))$ is $\mathrm{Pr}_\tau(\ulcorner f_\tau(A(\dot{x}_1, \ldots, \dot{x}_n))\urcorner)$.
- $\mathsf{QPL}_\tau(T)$
  $:= \{A \mid A$**: sentence and** $\forall f$**: arithmetical interpretation**$, T \vdash f_\tau(A)\}$

Theorem (Artemov, 1986)

For any Σ_1-sound theory T and Σ_1 definition $\tau_0(v)$ of T, there exists a Σ_1 definition $\tau_1(v)$ of T s.t. $\mathsf{QPL}_{\tau_0}(T) \nsubseteq \mathsf{QPL}_{\tau_1}(T)$.

Theorem (K., 2013)

Let $0 < i < j$.
There exists a Σ_1 definition $\tau_i(v)$ of some axiomatization of $\mathbf{I\Sigma_i}$ s.t. for any Σ_1 definition $\tau_j(v)$ of $\mathbf{I\Sigma_j}$,

$$\mathsf{QPL}_{\tau_i}(\mathbf{I\Sigma_i}) \nsubseteq \mathsf{QPL}_{\tau_j}(\mathbf{I\Sigma_j}) \text{ and } \mathsf{QPL}_{\tau_j}(\mathbf{I\Sigma_j}) \nsubseteq \mathsf{QPL}_{\tau_i}(\mathbf{I\Sigma_i}).$$

The situation of the inclusion relation between quantified provability logics is completely different from that of propositional case.

- **From Vardanyan's theorem, no recursively axiomatizable formal system characterizes $\mathsf{QPL}_\tau(T)$.**
- **Furthermore, the inclusion between quantified provability logics seems to be rarely established.**
- **From these circumstances, I investigated the inclusion relation between quantified provability logics in order to know more about the dependence of $\mathsf{QPL}_\tau(T)$ on T and $\mathrm{Pr}_\tau(x)$, and to better understand past researches.**

Outline

1. Background
2. Artemov's Lemma
3. Results

φ°

- **The main tool of my study is Artemov's Lemma used in the proof of Vardanyan's theorem.**
- **To state Artemov's Lemma, I prepare some definitions.**

Definition

- **We prepare predicate symbols** $P_Z(x)$, $P_S(x, y)$, $P_A(x, y, z)$, $P_M(x, y, z)$, $P_L(x, y)$ **and** $P_E(x, y)$ **corresponding to** 0, S, $+$, $\times$, $<$ **and** $=$, **respectively.**
- **For each** $\mathcal{L}_A$**-formula** φ, **let** φ^* **be a logically equivalent** $\mathcal{L}_A$**-formula where each atomic formula is one of the forms** $x = 0$, $S(x) = y$, $x + y = z$, $x \times y = z$, $x < y$ **and** $x = y$.
- **Let** φ° **be a relational formula obtained from** φ^* **by replacing each atomic formula with the corresponding relation symbol in** $\{P_Z, P_S, P_A, P_M, P_L, P_E\}$ **adequately.**
- **Then** φ° **is a quantified modal formula.**

For example, $(S(0) = x)^*$ **is** $\exists v(v = 0 \land S(v) = x)$ **and** $(S(0) = x)^\circ$ **is** $\exists v(P_Z(v) \land P_S(v, x))$.

Artemov's Lemma

Definition

Let D **be the modal sentence**

$$\bigwedge_{K \in \{Z,S,A,M,L,E\}} \Big(\forall \vec{x}(P_K(\vec{x}) \to \Box P_K(\vec{x})) \land \forall \vec{x}(\neg P_K(\vec{x}) \to \Box \neg P_K(\vec{x})) \Big).$$

Artemov's Lemma

There exists an $\mathcal{L}_A$-sentence ξ such that $\mathbf{I\Sigma_1} \vdash \xi$ and for any arithmetical interpretation f, Σ_1 definition $\tau(v)$ of T and $\mathcal{L}_A$-sentence φ,

$$\mathbf{I\Sigma_1} \vdash \mathrm{Con}_\tau \land f_\tau(\mathrm{D}) \land f_\tau(\xi^\circ) \to (\varphi \leftrightarrow f_\tau(\varphi^\circ)).$$

In the statement of the lemma, the $\mathcal{L}_A$-sentence ξ is a conjunction of several basic sentences of arithmetic such as $\forall x \exists y (S(x) = y)$ and $\forall x (x + 0 = x)$.

Visser and de Jonge's observation

What is important to me is the following consequence of Artemov's Lemma.

Proposition (Visser and de Jonge, 2006)

For any Σ_1 definition $\tau(v)$ of T and $\mathcal{L}_A$-sentence φ, TFAE:

1. $T + \mathrm{Con}_\tau \vdash \varphi$.
2. $\Diamond\top \wedge \mathrm{D} \wedge \xi^\circ \to \varphi^\circ \in \mathsf{QPL}_\tau(T)$.

Proof of Visser and de Jonge's proposition.

$(1 \Rightarrow 2)$: **Suppose** $T + \mathrm{Con}_\tau \vdash \varphi$.

By Artemov's Lemma, for any arithmetical interpretation f**,**

$$\mathbf{I\Sigma_1} \vdash \mathrm{Con}_\tau \wedge f_\tau(\mathrm{D}) \wedge f_\tau(\xi^\circ) \to (\varphi \leftrightarrow f_\tau(\varphi^\circ)).$$

Then $T \vdash \mathrm{Con}_\tau \wedge f_\tau(\mathrm{D}) \wedge f_\tau(\xi^\circ) \to f_\tau(\varphi^\circ)$.

$T \vdash f_\tau(\Diamond\top \wedge \mathrm{D} \wedge \xi^\circ \to \varphi^\circ)$.

Hence $\Diamond\top \wedge \mathrm{D} \wedge \xi^\circ \to \varphi^\circ \in \mathsf{QPL}_\tau(T)$.

$(2 \Rightarrow 1)$: **Suppose** $\Diamond\top \wedge \mathrm{D} \wedge \xi^\circ \to \varphi^\circ \in \mathsf{QPL}_\tau(T)$.

Let f **be an arithmetical interpretation such that for each** $K \in \{Z, S, A, M, L, E\}$, $f(P_K(\vec{x}))$ **is the intended** $\mathcal{L}_A$**-formula (for example,** $f(P_A(x, y, z))$ **is** $x + y = z$**).**

Then $\mathbf{I\Sigma_1} \vdash f_\tau(\mathrm{D}) \wedge f_\tau(\xi^\circ)$ **and** $\mathbf{I\Sigma_1} \vdash \varphi \leftrightarrow f_\tau(\varphi^\circ)$.

Since $T \vdash \mathrm{Con}_\tau \wedge f_\tau(\mathrm{D}) \wedge f_\tau(\xi^\circ) \to f_\tau(\varphi^\circ)$,

$T + \mathrm{Con}_\tau \vdash \varphi$. □

- **Visser and de Jonge's result shows that $\mathsf{QPL}_\tau(T)$ has the complete information about $\mathsf{Th}(T + \mathrm{Con}_\tau)$.**
- **Moreover, the following corollary concerning inclusions between quantified provability logics is important.**

Corollary

If $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, then $\mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0}) \subseteq \mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1})$.

Proof.

Suppose $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$.
Let φ be any $\mathcal{L}_A$-sentence with $T_0 + \mathrm{Con}_{\tau_0} \vdash \varphi$.
$\Diamond\top \wedge \mathrm{D} \wedge \xi^\circ \to \varphi^\circ \in \mathsf{QPL}_{\tau_0}(T_0)$. **(by Proposition)**
$\Diamond\top \wedge \mathrm{D} \wedge \xi^\circ \to \varphi^\circ \in \mathsf{QPL}_{\tau_1}(T_1)$. **(by the supposition)**
$T_1 + \mathrm{Con}_{\tau_1} \vdash \varphi$. **(by Proposition)**

□

Outline

Main theorem 1

Inspired by Visser and de Jonge's proposition, I investigated further consequences of inclusions between quantified provability logics that result from Artemov's Lemma.

Theorem (K.)

Let $\tau_0(v)$ **and** $\tau_1(v)$ **be** Σ_1 **definitions of** T_0 **and** T_1**, respectively.**
Suppose $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$**.**
Then:

1. $T_1 \vdash \mathrm{Con}^n_{\tau_0} \leftrightarrow \mathrm{Con}^n_{\tau_1}$ **for any** $n \geq 1$**;**
2. $\mathsf{Th}(T_0) \cap \Sigma_1 \subseteq \mathsf{Th}(T_1) \cap \Sigma_1$**;**
3. **for any** $\mathcal{L}_A$**-sentence** φ**,**

$$T_1 \vdash \mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi \urcorner);$$

4. **for any** Π_1**-sentence** φ**,**

$$T_1 \vdash \mathrm{Pr}_{\tau_1}(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi \urcorner).$$

Main theorem 1

Theorem (K.)

Let $\tau_0(v)$ **and** $\tau_1(v)$ **be** Σ_1 **definitions of** T_0 **and** T_1**, respectively. Suppose** $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$ **and** $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$**. Then:**

1. **for any** $\mathcal{L}_A$**-formula** $\varphi(\vec{x})$**,**
$$T_1 \vdash \forall \vec{x} \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi(\dot{\vec{x}}) \urcorner) \right);$$
2. **then for any** Π_1**-formula** $\varphi(\vec{x})$**,**
$$T_1 \vdash \forall \vec{x} (\mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner));$$
3. $\mathsf{QPL}_{\tau_0 + \mathrm{Con}^n_{\tau_0}}(T_0 + \mathrm{Con}^n_{\tau_0}) \subseteq \mathsf{QPL}_{\tau_1 + \mathrm{Con}^n_{\tau_1}}(T_1 + \mathrm{Con}^n_{\tau_1})$ **for any** $n \geq 1$**.**

Corollaries (1/3)

From this theorem, I obtained several refinements of known results.

Corollary 1 (A refinement of Montagna's theorem)

If $T_1 \nvdash \neg \mathrm{Con}_{\tau_1}$ **and** $T_0 \vdash \mathrm{Con}_{\tau_1} \to \mathrm{Con}^2_{\tau_0}$,
then $\mathsf{QPL}_{\tau_0}(T_0) \nsubseteq \mathsf{QPL}_{\tau_1}(T_1)$.

Proof.

Suppose $T_0 \vdash \mathrm{Con}_{\tau_1} \to \mathrm{Con}^2_{\tau_0}$ **and** $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$.
Then $T_1 \vdash \mathrm{Con}_{\tau_1} \to \mathrm{Con}^2_{\tau_0}$ **and** $T_1 \vdash \mathrm{Con}^2_{\tau_0} \leftrightarrow \mathrm{Con}^2_{\tau_1}$.
So $T_1 \vdash \mathrm{Con}_{\tau_1} \to \mathrm{Con}^2_{\tau_1}$.
By Löb's theorem, $T_1 \vdash \neg \mathrm{Con}_{\tau_1}$. □

Theorem (Montagna, 1984), restated

If T_1 **is finitely axiomatizable,** $T_1 \nvdash \neg \mathrm{Con}_{T_1}$ **and** $T_0 \vdash \mathrm{Con}_{T_1} \to \mathrm{Con}^2_{T_0}$,
then $\mathsf{QPL}(T_0) \nsubseteq \mathsf{QPL}(T_1)$.

Corollaries (2/3)

Corollary 2 (A refinement of Artemov's theorem)

For any Σ_1-sound theory T and Σ_1 definition $\tau_0(v)$ of T, there exists a Σ_1 definition $\tau_1(v)$ of T s.t. $\mathsf{QPL}_{\tau_0}(T) \nsubseteq \mathsf{QPL}_{\tau_1}(T)$ and $\mathsf{QPL}_{\tau_1}(T) \nsubseteq \mathsf{QPL}_{\tau_0}(T)$.

Proof.

Let $\tau_0(v)$ be any Σ_1 definition of T.
Since T is Σ_1-sound, it is known that there exists a Σ_1 definition $\tau_1(v)$ of T such that $T \nvdash \mathrm{Con}_{\tau_0} \to \mathrm{Con}_{\tau_1}$.
By the theorem, $\mathsf{QPL}_{\tau_0}(T) \nsubseteq \mathsf{QPL}_{\tau_1}(T)$ and $\mathsf{QPL}_{\tau_1}(T) \nsubseteq \mathsf{QPL}_{\tau_0}(T)$. □

Theorem (Artemov, 1986), restated

For any Σ_1-sound theory T and Σ_1 definition $\tau_0(v)$ of T, there exists a Σ_1 definition $\tau_1(v)$ of T s.t. $\mathsf{QPL}_{\tau_0}(T) \nsubseteq \mathsf{QPL}_{\tau_1}(T)$.

Corollaries (3/3)

Corollary 3

Suppose that T_0 is consistent, T_1 is Σ_1-sound and there exists a Σ_1 definition $\sigma_0(v)$ of T_0 such that $T_1 \vdash \mathrm{Rfn}_{\sigma_0}(\Sigma_1)$. Then, for any respective Σ_1 definitions $\tau_0(v)$ and $\tau_1(v)$ of T_0 and T_1, $\mathsf{QPL}_{\tau_0}(T_0) \nsubseteq \mathsf{QPL}_{\tau_1}(T_1)$ and $\mathsf{QPL}_{\tau_1}(T_1) \nsubseteq \mathsf{QPL}_{\tau_0}(T_0)$.

Example (A refinement my previous result)

Let $0 < i < j$.
For any respective Σ_1 definitions $\tau_i(v)$, $\tau_j(v)$ of $\mathbf{I\Sigma_i}$ and $\mathbf{I\Sigma_j}$,

$$\mathsf{QPL}_{\tau_i}(\mathbf{I\Sigma_i}) \nsubseteq \mathsf{QPL}_{\tau_j}(\mathbf{I\Sigma_j}) \text{ and } \mathsf{QPL}_{\tau_j}(\mathbf{I\Sigma_j}) \nsubseteq \mathsf{QPL}_{\tau_i}(\mathbf{I\Sigma_i}).$$

Theorem (K., 2013), restated

Let $0 < i < j$.
There exists a Σ_1 definition $\tau_i(v)$ of some axiomatization of $\mathbf{I\Sigma_i}$ s.t. for any Σ_1 definition $\tau_j(v)$ of $\mathbf{I\Sigma_j}$

$$\mathsf{QPL}_{\tau_i}(\mathbf{I\Sigma_i}) \nsubseteq \mathsf{QPL}_{\tau_j}(\mathbf{I\Sigma_j}) \text{ and } \mathsf{QPL}_{\tau_j}(\mathbf{I\Sigma_j}) \nsubseteq \mathsf{QPL}_{\tau_i}(\mathbf{I\Sigma_i}).$$

Σ_1 provability logics

Researches on restricted arithmetical interpretations have also been done by many authors.

Definition (Σ_1 arithmetical interpretation)

An arithmetical interpretation f is called Σ_1 if

- **(Propositional case) for any propositional variable p, $f(p)$ is a Σ_1 sentence;**
- **(Predicate case) for any atomic formula $P(\vec{x})$, $f(P(\vec{x}))$ is a Σ_1 formula.**

Definition (Σ_1 provability logics)

- $\mathsf{PL}^{\Sigma_1}(T) := \{A \mid \forall f : \Sigma_1$ **arithmetical interpretation**$, T \vdash f_T(A)\}$
- $\mathsf{QPL}^{\Sigma_1}(T)$

  $:= \{A \mid A$ **is a sentence and** $\forall f : \Sigma_1$ **arithmetical interpretation**$, T \vdash f_T(A)\}$
- $\mathsf{QPL}^{\Sigma_1}_{\tau}(T)$

  $:= \{A \mid A$ **is a sentence and** $\forall f : \Sigma_1$ **arithmetical interpretation**$, T \vdash f_\tau(A)\}$

Known results for Σ_1 provability logics

In the propositional case, $\mathsf{PL}^{\Sigma_1}(T)$ is recursively axiomatizable.

Theorem (Visser)

If T is Σ_1-sound, then $\mathsf{PL}^{\Sigma_1}(T)$ is characterized by a formal system GLV.

In the predicate case, an analogue of Vardanyan's theorem holds.

Theorem (Berarducci, 1989)

$\mathsf{QPL}^{\Sigma_1}(\mathbf{PA})$ is Π^0_2-complete.

However, there is some benefit to deal with Σ_1 arithmetical interpretations in my study.

- **In the proof of Artemov's Lemma, the sentence $\mathrm{Con}_\tau \wedge f_\tau(\mathrm{D})$ is used to make the formulas $f(P_K(\vec{x}))$ and $\neg f(P_K(\vec{x}))$ equivalent to Σ_1 formulas for each $K \in \{Z, S, A, M, L, E\}$:**

$$f_\tau(P_K(\vec{x})) \leftrightarrow \mathrm{Pr}_\tau(\ulcorner f_\tau(P_K(\dot{\vec{x}}))\urcorner)$$

$$\neg f_\tau(P_K(\vec{x})) \leftrightarrow \mathrm{Pr}_\tau(\ulcorner \neg f_\tau(P_K(\dot{\vec{x}}))\urcorner).$$

- **In the case that f is a Σ_1 arithmetical interpretation, the same result holds without assuming $\mathrm{Con}_\tau \wedge f_\tau(\mathrm{D})$ by adding sufficiently many theorems of $\mathbf{I\Sigma_1}$ to the sentence ξ as conjuncts.**
- **This is guaranteed by the following equivalences:**
  - $\neg P_Z(x) \leftrightarrow \exists y P_S(y, x)$;
  - $\neg P_S(x, y) \leftrightarrow \exists z(P_S(x, z) \wedge (P_L(z, y) \vee P_L(y, z)))$;
  - $\cdots$
  - $\neg P_E(x, y) \leftrightarrow P_L(x, y) \vee P_L(y, x)$.

Artemov's Lemma w.r.t. Σ_1 arithmetical interpretations

Then I obtained the following version of Artemov's Lemma with respect to Σ_1 arithmetical interpretations.

Lemma (K.)

There exists an $\mathcal{L}_A$-sentence ξ such that $\mathbf{I\Sigma_1} \vdash \xi$ and for any Σ_1 arithmetical interpretation f, Σ_1 definition $\tau(v)$ of T and any $\mathcal{L}_A$-sentence φ,

$$\mathbf{I\Sigma_1} \vdash f_\tau(\xi^\circ) \to (\varphi \leftrightarrow f_\tau(\varphi^\circ)).$$

Main theorem 2

By using this lemma, I proved the following theorem.

Theorem (K.)

Let $\tau_0(v)$ and $\tau_1(v)$ be Σ_1 definitions of T_0 and T_1, respectively. TFAE:

1. $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$.
2. $\mathsf{Th}(T_0) \subseteq \mathsf{Th}(T_1)$ **and for any $\mathcal{L}_A$-formula $\varphi(\vec{x})$,**

$$T_1 \vdash \forall \vec{x}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)).$$

Corollary

If $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$, then $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$.

Corollary and Problem

Conlusion

- By investigating several conclusions of the inclusion $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, I showed that $\mathsf{QPL}_{\tau}(T)$ really depends on T and $\mathrm{Pr}_{\tau}(x)$, and that the inclusion $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$ rarely hold.
- By providing a necessary and sufficient condition for the inclusion $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$, I found an order in the world of quantified provability logics.

Problem

Can we characterize the relation $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$?

References

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