Incompleteness theorems for weak theories of arithmetic and some stronger versions of the incompleteness theorem
(a survey)

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Overview

Preliminaries

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where $\ulcorner \phi \urcorner$ denotes the numeral expressing the Gödel number of ϕ and Pr_T is a Σ_1-formula expressing provability in T

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  where $\ulcorner \phi \urcorner$ denotes the numeral expressing the Gödel number of ϕ and Pr_T is a Σ_1-formula expressing provability in T
- the 2nd Incompleteness Theorem says that Con_T, which is $\neg Pr_T(0 = 1)$, is not provable in any sufficiently strong T
- it is proved by proving

$$T \;\vdash\; Con_T \to \gamma_T$$

Question 1. Does the 2nd Incompleteness Theorem hold for weak theories?

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Question 2. Are there stronger versions?

Definable cuts

Definition

A formula $I(x)$ is a *cut in* T if T proves

1. $I(0)$
2. $I(x) \to I(S(x))$
3. $I(x) \land y < x \to I(y)$

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3. $I(x) \wedge y < x \rightarrow I(y)$

Lemma (R. Solovey)

Given a cut I in T, one can construct a cut J such that T proves

1. $J(x) \rightarrow I(x)$,
2. $J(x) \wedge J(y) \rightarrow J(x + y)$
3. $J(x) \wedge J(y) \rightarrow J(x \cdot y)$

Proof.

- $J_1(x) \equiv \forall y(I(y) \rightarrow I(y+x))$
- $J(x) \equiv \forall y(J_1(y) \rightarrow J_1(y \cdot x))$

□

Proof.

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- $J(x) \equiv \forall y(J_1(y) \to J_1(y \cdot x))$

$\square$

We can go on and construct $J(x)$ such that

- $T \vdash J(x) \to J(x^{\log\lceil x+1 \rceil})$

and more, but not $J(x) \to J(2^x)$.

Proof.

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and more, but not $J(x) \to J(2^x)$.
Instead we have

Lemma

Given a cut $I(x)$ in T, one can construct a cut J such that T proves

- $J(x) \to I(2^x)$.

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Given a cut $I(x)$ in T, one can construct a cut $J(x)$ which is provably closed on $+$ and $\cdot$ and moreover satisfies induction for all Δ_0 formulas.

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This theorem is provable in $I\Delta_0 + \Omega_1$, where $\Omega_1 := \forall x \exists y = x^{\log\lceil x+1 \rceil}$, hence we get

Corollary

$I\Delta_0 + \Omega_1 \vdash Con_{I\Delta_0} \equiv Con_Q$

A strengthening of the 2nd Incompleteness Theorem

Theorem

For every cut $I(x)$ in T it is consistent with T that there exists a proof of contradiction in T whose Gödel number g satisfies $I(g)$.

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Proof idea.

- Given I we can define J such that, provably in T, $J \subseteq I$ and J satisfies $I\Delta_0$.
- We can imagine that the natural numbers of T are elements x that satisfy $J(x)$ and prove the 2nd Incompleteness Theorem with this concept of numbers.
- Hence it is consistent that the Gödel number of contradiction is in $J \subseteq I$.

□

Corollary

For any theory T extending Q,

1. *the statement of the 2nd Incompleteness Theorem is meaningful*
2. *and the theorem holds true.*

An application

Let *Exp* denote an axiom saying $\forall x \exists y = 2^x$.

Theorem (J. Paris and A. Wilkie)

$I\Delta_0 + Exp$ *does not prove* Con_Q.

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Proof sketch.

Let 2^x_k denote k-times iterated exponential function 2^x.

Lemma

Let $\phi(x) \in \Delta_0$. Then

$$I\Delta_0 + Exp \vdash \forall x \phi(x)$$

iff there exists k such that

$$I\Delta_0 \vdash \exists y = 2^x_k \to \phi(x).$$

Suppose $I\Delta_0 + Exp \vdash Con_Q$. Then it also proves $Con_{I\Delta_0}$. Let k be such that

$$I\Delta_0 \vdash \exists y = 2^x_k \to Con_{I\Delta_0} \qquad (1)$$

By a previous lemma iterated k-times, there exists a cut $I(x)$ in $I\Delta_0$ such that

$$I\Delta_0 \vdash I(x) \to \exists y = 2^x_k.$$

By the strengthening of the 2nd Incompleteness Theorem, it is consistent with $I\Delta_0$ that there exists a proof of contradiction x in $I\Delta_0$ such that $I(x)$ holds true. But this contradicts to (1). □

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Theorem
$I\Delta_0 + \Omega_1 + Con_Q$ proves all Π_1 theorems of $I\Delta_0 + Exp$.

- Hence Con_Q is *stronger than all Π_1 theorems of $I\Delta_0 + Exp$.*

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- This can be formalized in $I\Delta_0 + \Omega_1$, so we have

$$I\Delta_0 + \Omega_1 \vdash \; Pr_{I\Delta_0}(\forall x(I(x) \to \phi(x))).$$

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by formalized Σ-completeness.

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- $I\Delta_0 + \Omega_1 + Con_{I\Delta_0} \vdash \forall x \ \phi(x)$.
- $I\Delta_0 + \Omega_1 \vdash Con_{I\Delta_0} \equiv Con_Q$ by formalizing Wilkie's Theorem.

□

The finite version of the 2nd Incompleteness Theorem

Theorem (H. Friedman)

Let $Con_T(x)$ be a formula saying:

"there is no proof of contradiction in T of length $\leq x$".

Then

any T-proof of the sentence $Con_T(\bar{n})$ has length $\geq n^{1-o(1)}$.

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any T-proof of the sentence $Con_T(\bar{n})$ has length $\geq n^{1-o(1)}$.

- $\bar{n}$ is a binary numeral, i.e., if $n = \sum a_i 2^i$, a_i zeros and ones, then $\bar{n}$ is the term

$$\bar{a}_1 + \bar{2} \cdot (\bar{a}_2 + \bar{2}(\dots \quad + \bar{a}_n) \dots)$$

  whose length is $O(\log n)$
- the provability predicate is expressed by a Σ^b_1 formula
- we are still assuming that T is consistent

Proof-idea

- Define the self-referential formulas γ_m

$$\gamma_m \equiv \neg LPrf_T(\bar{m}, \ulcorner \gamma_m \urcorner)$$

where $LPrf_T(x, y)$ says that y has a T-proof of length $\leq x$.

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where $LPrf_T(x, y)$ says that y has a T-proof of length $\leq x$.

- Prove that γ_m does not have proofs of length $< m$.
- Prove that $Con_T(n) \rightarrow \gamma_m$, for $m = n^{1-o(1)}$, has a proof of length $(\log n)^{O(1)}$.

□

Some applications

Proof of the 2nd Incompleteness Theorem.

Suppose T proves $\forall x\ Con_T(x)$. Then every instance $Con_T(\bar{n})$ has a proof of length $O(\log n)$ — contradiction. □

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Let $I(x)$ be a cut in T and suppose T proves $\forall x(I(x) \to Con_T(x))$.

Let $J(x)$ be a cut in T such that T proves $J \subseteq I$ and J is closed under $+$ and $\cdot$. Then for every n we have a T-proof of length $O(\log n)$ of $J(\bar{n})$, hence also $I(\bar{n})$.

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From this, we get an $O(\log n)$-length proof of $Con_T(\bar{n})$ as above — contradiction. $\square$

Bounded consistency

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It is not possible to separate the fragments using $Con_{S_2^i}$, because $S_2 \nvdash Con_Q$. Therefore he proposed *bounded consistency*.

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$BDCon_T$ is a formalization of "there is no T-proof of contradiction that only uses *bounded formulas*".

He proved that there is at most one i such that $S_2^{i+1} \vdash BDCon_{S_2^i}$, but in fact, there is no such i.

Theorem
$S_2 \nvdash BDCon_{S_2^1}$.

Theorem
$S_2 \nvdash BDCon_{S^1_2}$.

Proof-idea.

- Prove a lower bound n^{ϵ}, $\epsilon > 0$ on the lengths of S^1_2-proofs of $BDCon_{S^1_2}(\bar{n})$.
- Show that proofs of such sentences in S^1_2 are only polynomially longer than in S_2.

Theorem

$S_2 \nvdash BDCon_{S_2^1}$.

Proof-idea.

- Prove a lower bound n^ϵ, $\epsilon > 0$ on the lengths of S_2^1-proofs of $BDCon_{S_2^1}(\bar{n})$.
- Show that proofs of such sentences in S_2^1 are only polynomially longer than in S_2.
- Thus we get an n^δ, $\delta > 0$ lower bound on the lengths of S_2-proofs of $BDCon_{S_2^1}(\bar{n})$.
- The rest is the same as in the proof of the 2nd Incompleteness Theorem.

□

Another strengthening of the 2nd Incompleteness Theorem

Let $\kappa(x)$ be a formula with one free variable x such that T proves

1. $\kappa(x) \wedge y \leq x \rightarrow \kappa(y)$,
2. $\kappa(\bar{0}), \kappa(\bar{1}), \kappa(\bar{2}) \ldots.$

Is it consistent with T to assume that there is a proof of contradiction in κ?

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Of course, not! Define

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$$\kappa(x) \equiv \forall y \leq x \neg Prf_T(y, \perp)$$

It cannot even be close to κ:

$$\kappa(x) \equiv \forall y \leq f(x) \neg Prf_T(y, \perp)$$

where f is any definable function.
Yet ...

Theorem

Let $T \supseteq I\Delta_0$ be a consistent theory. Let $\kappa(x)$ be a bounded formula and let h be any Δ_0-definable function such that $h(n) \to \infty$.[2] *Suppose T proves*[3]

$$\kappa(\bar{0}), \kappa(\bar{1}), \kappa(\bar{2}) \ldots$$

Then

$$\exists x \exists y \leq 2^{x^{h(x)}}(\kappa(x) \wedge Prf_T(y, \perp))$$

is consistent with T.

[2] provably in T, but I think that h may be a constant depending on K and T.

[3] Since $\kappa \in \Delta_0$, this is equivalent to $\mathbb{N} \models \kappa(\bar{0}), \kappa(\bar{1}), \ldots$.

Injecting inconsistencies into models

Theorem (J.K. and P.P., improving results of Paris and Dimitracopoulos, Hájek, and Solovay)

Let M be a countable nonstandard model of T with nonstandard elements $h < a$. Then there exists a countable model N of T such that

1. $M \cap [0, a] = N \cap [0, a]$,
2. $N \models \exists y \leq 2^{a^h}\ Prf_T(y, \bot)$.

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Assume w.l.o.g. that in all bounded quantifiers of $\kappa(x)$ the bounding term is x.

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Let M be a countable nonstandard model of T. By overspill, there exists a nonstandard number a such that $M \models \kappa(a)$. Let $h := h(a)$.

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Let N be the model from the above theorem. Note that $N \models \kappa(a)$.

□

References

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Thank you