Gödel in AProS

Wilfried Sieg

Carnegie Mellon University

Prelude.

My talk is not primarily focused on the presentation of new mathematical or metamathematical results; rather, it is describing work that I have done over much of my professional life and that seems, prima facie, to concern disconnected topics. For example,

automated search for normal natural deduction proofs;

logical calculi that allow reasoning with gaps;

Prelude.

formal proofs of the Incompleteness Theorems;

editing Gödel’s correspondence with Herbrand, Nagel, Post, and von Neumann for Volume V of his *Collected Works*;

developing web-based classes for basic logic, elementary set theory, and incompleteness results.

Prelude.

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developing web-based classes for basic logic, elementary set theory, and incompleteness results.

In retrospect, there were three central, mostly *educational motivations*:

Prelude.

(**1**) to provide beginning students a space where they can explore, intelligently supported, the construction of proofs and of counterexamples;

(**2**) to show them a methodological framework for mathematics and the formalization of mathematical practice in that frame;

(**3**) to explore with them a jewel of mathematical, logical, and philosophical reflection, namely, the incompleteness theorems and the accompanying analysis of computability.

Prelude.

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Research Associate in Patrick Suppes' IMSSS; *CAI*.

My Task: develop a computer-based course in *elementary proof theory* starting with a presentation of Gödel's Incompleteness Theorems for ZF.

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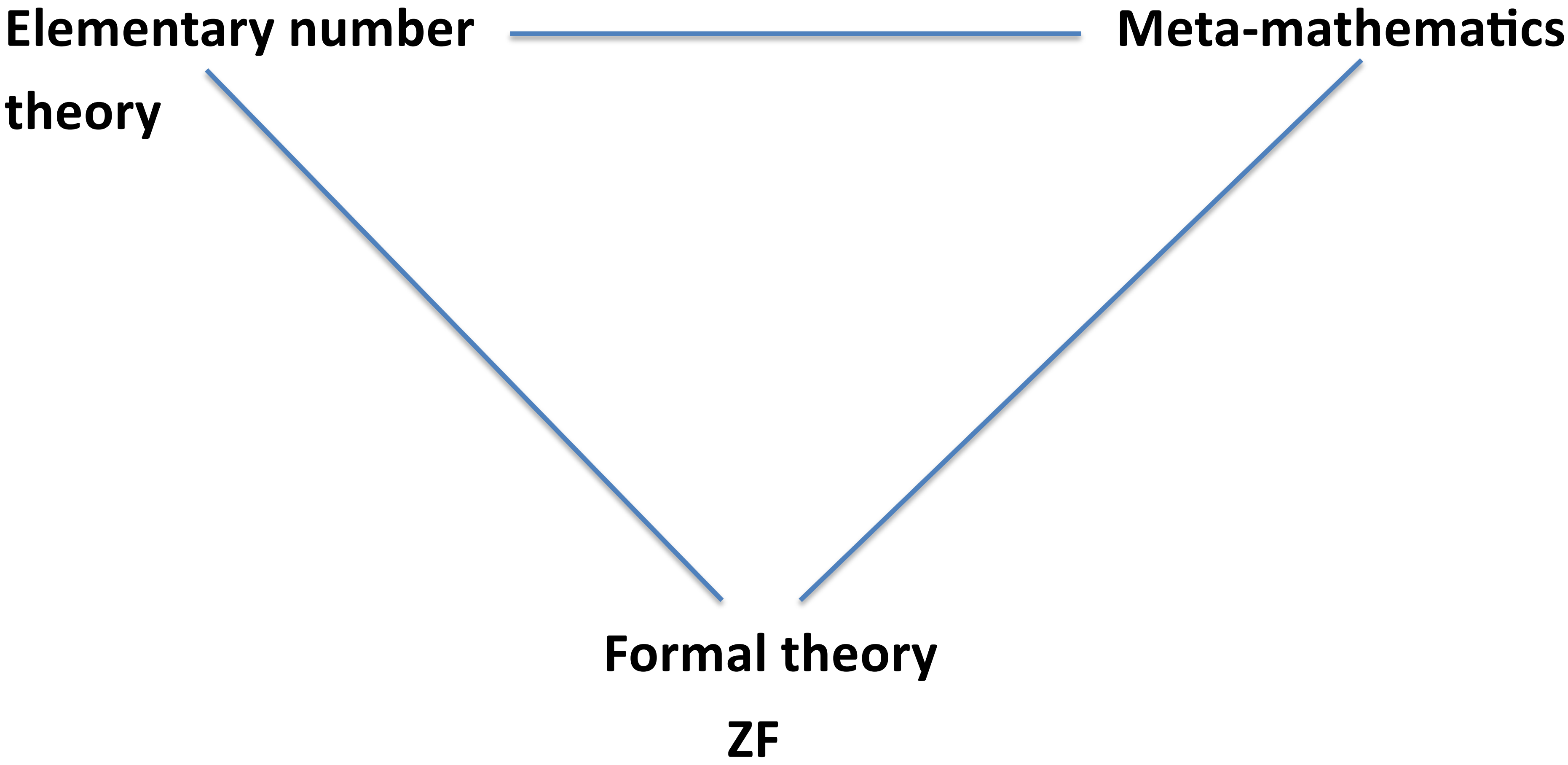
Stage 1. Direct representation and verification in EXCHECK.

Stage 1.

At first, when thinking about that task, I was almost convinced that it was practically infeasible. Just think of the standard triangle reflecting the arithmetization of meta-mathematics.

Stage 1.


Elementary number theory
Meta-mathematics
Formal theory
ZF

Stage 1.

How could this threefold proliferation of notions and operations, thus of necessary notations, be reflected on a tiny computer screen - with utmost formal precision and yet intelligibly?

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When answering the question, “What is the central insight underlying the proofs of Gödel’s theorems?” by “It is *the internalization of syntax*!”, it occurred to me that that can be achieved by a *direct* representation of syntactic objects, notions and operations.

Stage 1.

After all, the latter are defined in meta-mathematics by elementary inductive definitions and structural recursion, respectively.

TEM: syntactic objects – binary trees; basic axioms for pairing and projections; induction. In addition, closure and minimality axioms for the inductively defined notions like “formula” and “proof”; recursion equations for the needed functions like substitution.

Stage 1.

Using Dedekindian techniques, they can be explicitly defined in ZF and shown to satisfy the crucial representability conditions. That set the stage for the formal verification of the incompleteness theorems and related ones (like Löb's theorem) for ZF. [1]

The verification was to be done in the theorem proving system EXCHECK that had been developed by Suppes and collaborators. [2]

Stage 1.

When I left Stanford in 1977 to start teaching at Columbia, I also left this work that had not been fully completed. In 1980 I wrote a brief report on its status with Ingrid and Sten Lindstrom who continued work on this Gödel project. [3]

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I returned to it around 15 years later and did work that led to my 2005 paper with Clinton Field; the paper was entitled *Automated search for Gödel's proofs.* [4] The basic meta-mathematical set-up in this paper is exactly that of my earlier Stanford work.

Stage 1.

You might wonder, why I went back to the Gödel work. What had changed during the intervening 15 years? Why attempt to find the proofs in an automated, abstract way?

It was a natural extension of another project that had been started in 1986 shortly after I had joined the faculty at Carnegie Mellon.

Interlude 1. Proof Tutor project and automated search for normal proofs.

Interlude 1.

Carnegie Mellon's philosophy department had preserved a computer-based introduction to logic; it was called VALID and had been developed at Stanford by Suppes already in the late 1960s.

However, it was judged to teach the construction of proofs not very well, indeed, quite poorly. So, the question was, how to make VALID's teaching more effective.

Interlude 1.

The answer: supplement VALID with an intelligent automated **PROOF TUTOR**! What was the idea? …

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Let me make a brief excursion on the history of ND calculi to make very clear why ND calculi are not just ordinary logical calculi, but are rather significant for a particular perspective on proof theory.

Interlude 1.

H&B’s axiomatic calculus:

&I(ntroduction)	$\phi \to (\psi \to (\phi \,\&\, \psi))$
&E(limination)	$(\phi \,\&\, \psi) \to \phi$ and $(\phi \,\&\, \psi) \to \psi$
∨I(ntroduction)	$\phi \to (\phi \vee \psi)$ and $\psi \to (\phi \vee \psi)$
∨E(limination)	$(\phi \vee \psi) \to ((\phi \to \chi) \to ((\psi \to \chi) \to \chi))$

Reasons for this shift away from the calculus of *Principia Mathematica*: practical and methodological; also, tree representation of proofs to carry out proof theoretic transformations.

Interlude 1.

Gentzen's ND

&I

$$\frac{\begin{matrix}\downarrow & \downarrow\\ \phi & \psi\end{matrix}}{\phi \,\&\, \psi}$$

&E

$$\frac{\begin{matrix}\downarrow\\ (\phi \,\&\, \psi)\end{matrix}}{\phi} \quad \text{and} \quad \frac{\begin{matrix}\downarrow\\ (\phi \,\&\, \psi)\end{matrix}}{\psi}$$

∨I

$$\frac{\begin{matrix}\downarrow\\ \phi\end{matrix}}{(\phi \vee \psi)} \quad \text{and} \quad \frac{\begin{matrix}\downarrow\\ \psi\end{matrix}}{(\phi \vee \psi)}$$

∨E

$$\frac{\begin{matrix} & [\phi] & [\psi]\\ \downarrow & \downarrow & \downarrow\\ (\phi \vee \psi) & \chi & \chi\end{matrix}}{\chi}$$

Interlude 1.

Distinctive for Gentzen were not the I- and E-rules, but rather making and discharging assumptions that reflected, in his view, a crucial feature of mathematical practice.

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After all, the basic goal of proof theory, according to both Hilbert and Gentzen, was to consider the proofs of mathematical practice as its objects and develop a "theory of the specifically mathematical proof"!

Interlude 1.

ND calculi do a very good job in capturing important structural features of ordinary mathematical reasoning. However, there are backward steps in such reasoning that ND calculi do NOT reflect in their syntactic configuration.

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What was desired then were ND-type calculi that allow forward and backward steps. So, we developed **NIC calculi** for classical and intuitionist sentential logic. [5] and [6] They were extended to first-order logic only a few years later. [7]

Interlude 1.

NIC stands for “natural intercalation”, as the basic idea is direct: close the gap between assumptions and goals by elimination steps in the forward direction and by inverted introduction steps in the backward direction.

Interlude 1.

Sequent formulation of ND (just for disjunctions) as given in (Gentzen 1936).

$$\vee\mathrm{I} \qquad \dfrac{\overset{\downarrow}{\Gamma \supset \phi}}{\Gamma \supset (\phi \vee \psi)} \quad \text{and} \quad \dfrac{\overset{\downarrow}{\Gamma \supset \psi}}{\Gamma \supset (\phi \vee \psi)}$$

$$\vee\mathrm{E} \qquad \dfrac{\overset{\downarrow}{\Gamma \supset (\phi \vee \psi)} \quad \overset{\downarrow}{\Gamma, \phi \supset \chi} \quad \overset{\downarrow}{\Gamma, \psi \supset \chi}}{\Gamma \supset \chi}$$

Interlude 1.

Some E-rules of NIC calculi using *structured sequents*:

$$\frac{\Gamma; \Delta, (\phi \,\&\, \psi), \phi \supset \chi}{\Gamma; \Delta, (\phi \,\&\, \psi) \supset \chi}\ \&E_1(\sigma) \qquad \frac{\Gamma; \Delta, (\phi \,\&\, \psi), \psi \supset \chi}{\Gamma; \Delta, (\phi \,\&\, \psi) \supset \chi}\ \&E_2(\sigma)$$

$$\frac{\Gamma, \phi; \supset \chi \quad \Gamma, \psi; \supset \chi}{\Gamma; \Delta, (\phi \vee \psi) \supset \chi}\ \vee E(\sigma[\vee]) \qquad \frac{\Gamma; \Delta, (\phi \to \psi), \psi \supset \chi \quad \Gamma; \supset \phi}{\Gamma; \Delta, (\phi \to \psi) \supset \chi} \to E$$

Interlude 1.

In any event, the development of the calculi and their meta-mathematical investigation went hand in hand with the implementation of a proof search system we now call ***AProS***.

The crucial meta-mathematical results in [8] and [9] were *strengthened completeness theorems* for both classical and intuitionist first-order logic in this form:

Interlude 1.

Either there is a counterexample to Γ, F or a *normal* proof from Γ to F. (NIC proofs are isomorphic to normal natural deduction proofs.)

(Not proved using a standard completeness proof and the normalize, but a direct proof – analogous to the completeness proof for the cut-free sequent calculus.)

That provided the theoretical basis for the AProS procedure … with three main strategic search steps: Extraction, Inversion, and Refutation.

Interlude 1.

Back to the question “Why attempting to find the proofs of the incompleteness theorems in an automated way?”

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Back to the question “Why attempting to find the proofs of the incompleteness theorems in an automated way?”

This question has a very simple answer: in early 2002, AProS was seen to be quite effective in finding good, strategic proofs and I wanted to see what the system could do beyond proving logical truths!

Stage 2. Automated search for Gödel’s proofs – at an abstract level.

Stage 2.

By proof at “an abstract level” I simply mean that certain facts are axiomatically taken for granted: in this case, representability and derivability conditions, but also, for example, the definitions of the Gödel and Löb sentences.

However, that worked only after we provided additional strategic guidance that reflects the intricate connection between meta-theoretic and object-theoretic reasoning.

Stage 2.

1.	$\mathsf{ZF}^* \vdash (G \leftrightarrow \neg(\mathsf{theo}(\boldsymbol{G})))$	Premise
2.	$\mathsf{ZF}^*\,\mathsf{CONS}$	Premise
3.	$\mathsf{ZF}^*\,\mathsf{CONS\ IFF\ NOT}\,(\mathsf{ZF}^* \vdash (G)\ \mathsf{AND}\ \mathsf{ZF}^* \vdash (\neg(G)))$	Premise
4.	$\mathsf{ZF}^* \vdash (G)$	Assumption
* 5.	G	Assumption
* 6.	$\mathsf{theo}(\boldsymbol{G})$	Rep 4
* 7.	$(G \leftrightarrow \neg(\mathsf{theo}(\boldsymbol{G})))$	Prov E 1
* 8.	$\neg(\mathsf{theo}(\boldsymbol{G}))$	Iff E R $7, 5$
* 9.	$\neg(G)$	Not I $5, 6, 8$
10.	$\mathsf{ZF}^* \vdash (\neg(G))$	Prov I 9
11.	$\mathsf{ZF}^* \vdash (G)\ \mathsf{AND}\ \mathsf{ZF}^* \vdash (\neg(G))$	And I $4, 10$
12.	$\mathsf{NOT}\,(\mathsf{ZF}^* \vdash (G)\ \mathsf{AND}\ \mathsf{ZF}^* \vdash (\neg(G)))$	Iff E R $3, 2$
13.	$\mathsf{NOT}\,(\mathsf{ZF}^* \vdash (G))$	Not I $4, 11, 12$ $\square$

Stage 2.

Interacting with the system was, however, very awkward: problems had to be entered in a special way; then the automated search took place; finally, the found proof could be rendered in roughly the above format.

At that point there was no interface in which one could actually construct proofs! That required yet another project that started around 2007.

Interlude 2. Web-based courses – interfaces for proof construction.

Interlude 2.

OLI ➔ *Logic & Proofs*. I started around 2007 to design and implement an introduction to modern symbolic logic with a focus on strategic construction of ND proofs and the definition of counterexamples from truth trees.

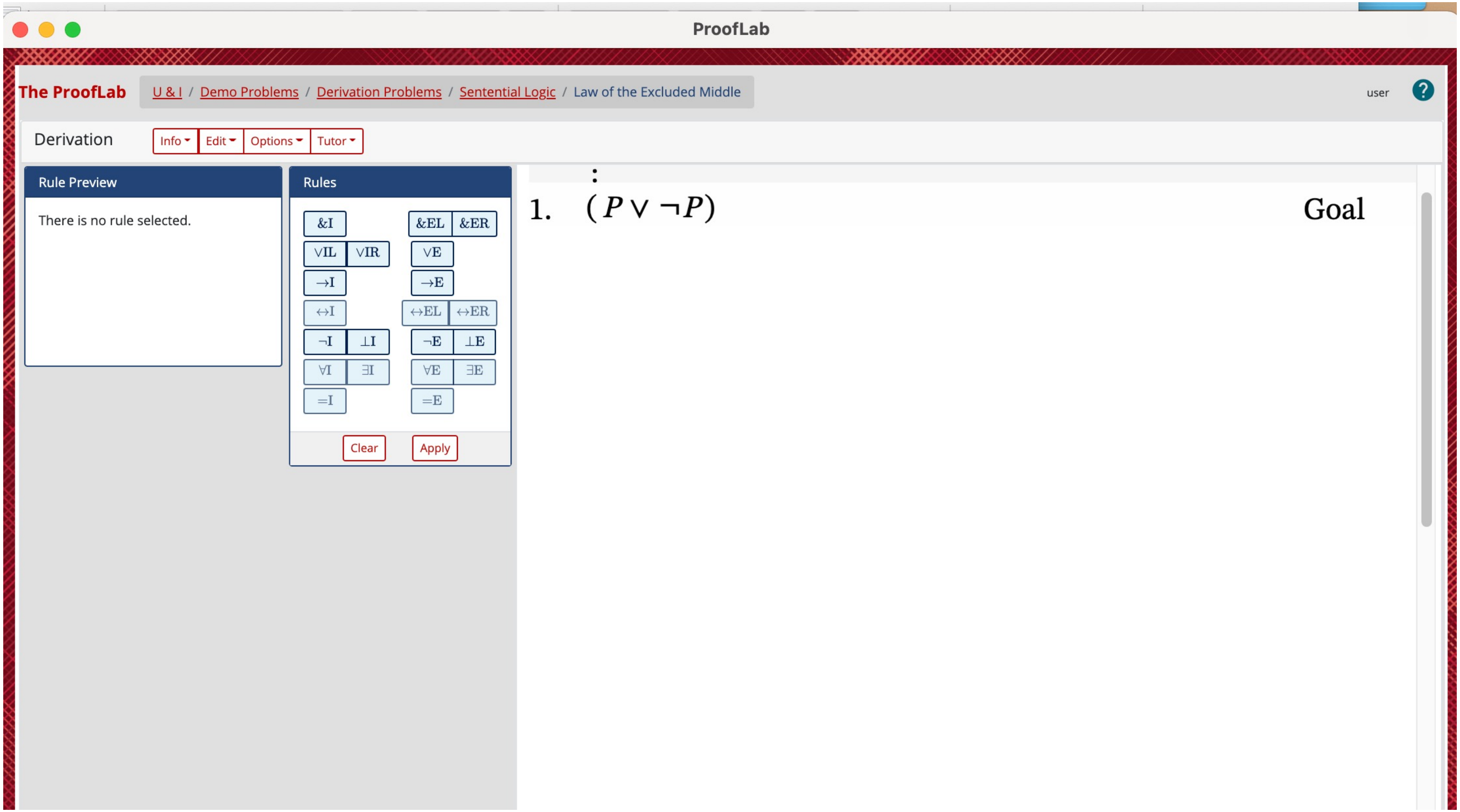
I can’t show you the result of an enormous amount of work that went into the course; so, I recommend a visit to *Logic & Proofs at* https://oli.cmu.edu/courses/logic-proofs-copy/

Interlude 2.

For continuing the overall story, the most important fact is that we had to design an interface for proof construction and we did this with bi-directional partial Fitch diagrams.

Here is the current version in which I constructed a proof of tertium non datur:

Interlude 2.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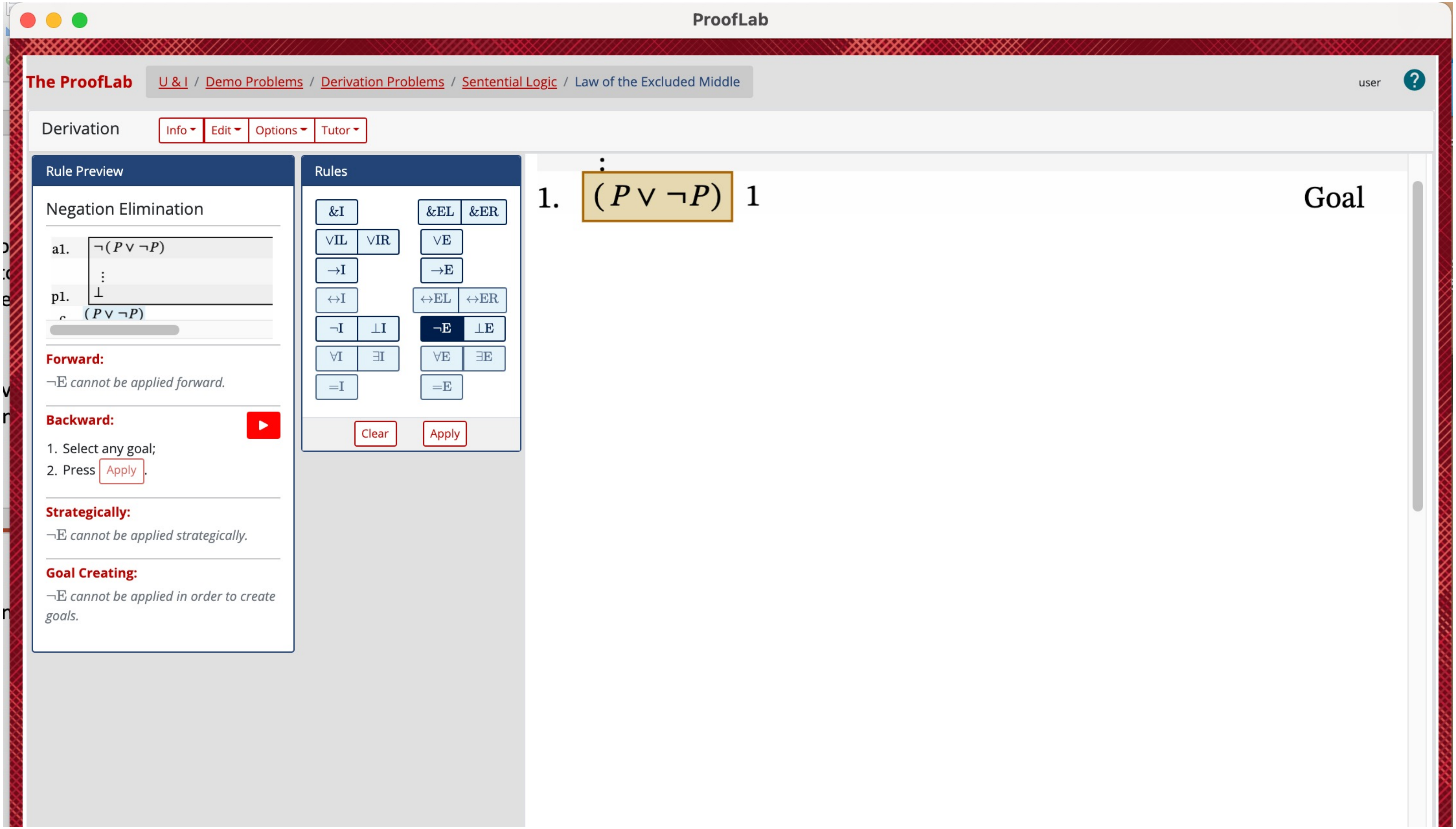
Apply
1. (P ∨ ¬P)
Goal

Interlude 2.



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Negation Elimination
a1. ¬(P ∨ ¬P)
p1. ⊥
(P ∨ ¬P)
Forward:
¬E cannot be applied forward.
Backward:
1. Select any goal;
2. Press Apply.
Strategically:
¬E cannot be applied strategically.
Goal Creating:
¬E cannot be applied in order to create goals.
Rules
&I
&EL
&ER
∨IL
∨IR
∨E
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→E
↔I
↔EL
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¬I
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¬E
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Clear
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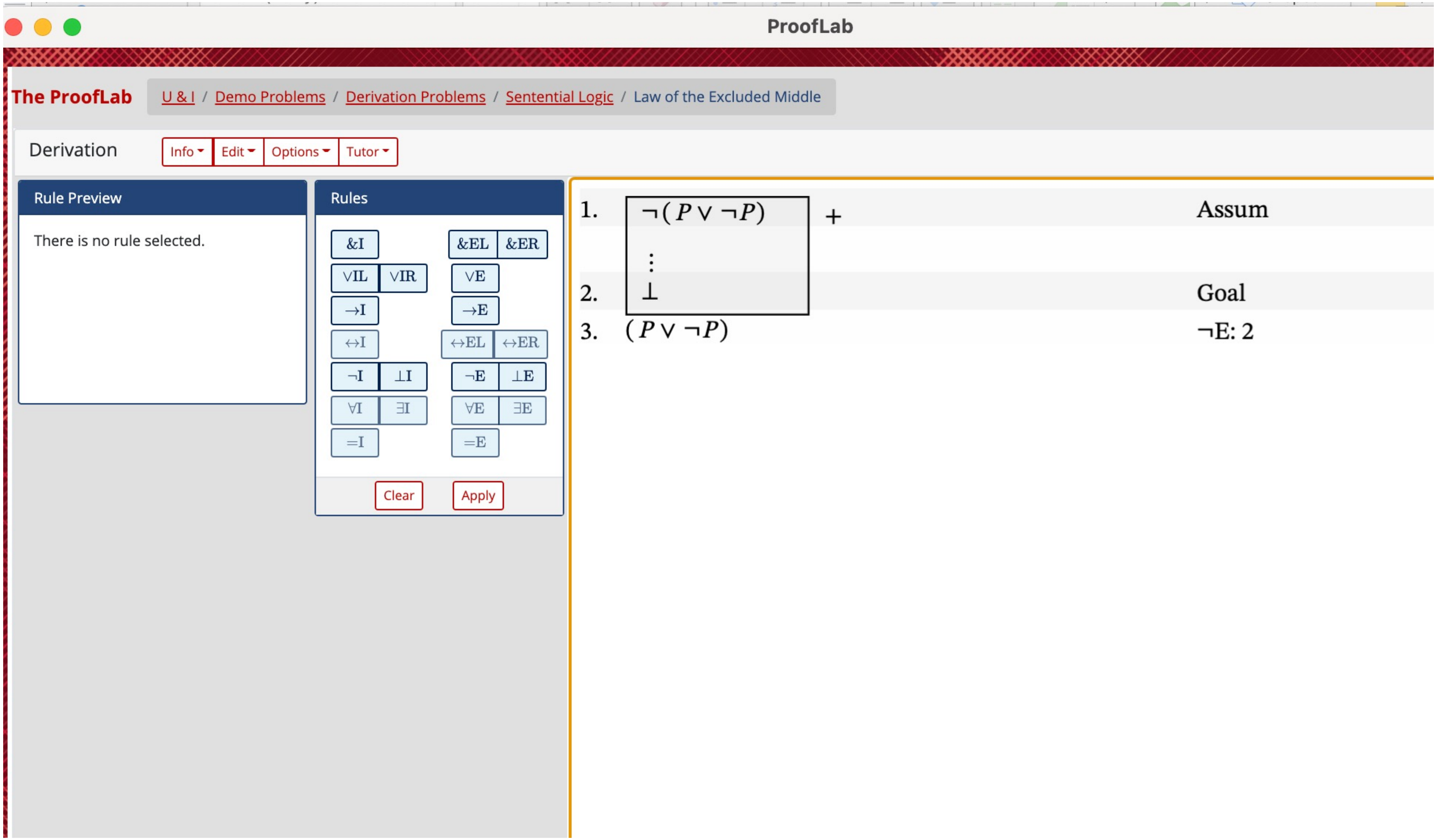
1. (P ∨ ¬P) 1
Goal

Interlude 2.



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∀I
∃I
∀E
∃E
=I
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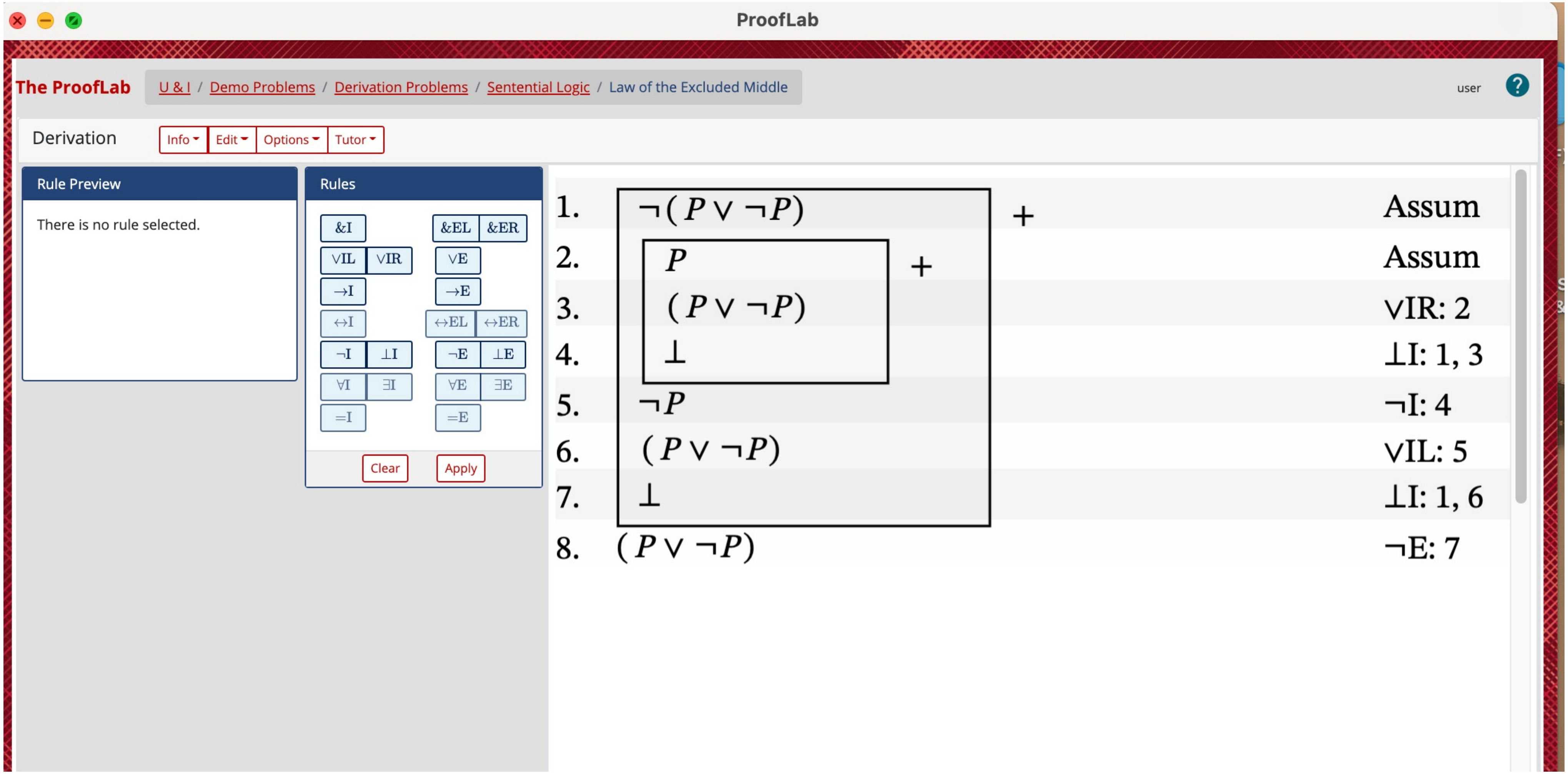
Clear
Apply
1. ¬(P ∨ ¬P) Assum
⋮
2. ⊥ Goal
3. (P ∨ ¬P) ¬E: 2

Interlude 2.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Rules
&I &EL &ER
∨IL ∨IR ∨E
→I →E
↔I ↔EL ↔ER
¬I ⊥I ¬E ⊥E
∀I ∃I ∀E ∃E
=I =E
Clear Apply
1. ¬(P ∨ ¬P) Assum
2. P Assum
3. (P ∨ ¬P) ∨IR: 2
4. ⊥ ⊥I: 1, 3
5. ¬P ¬I: 4
6. (P ∨ ¬P) ∨IL: 5
7. ⊥ ⊥I: 1, 6
8. (P ∨ ¬P) ¬E: 7

Interlude 2.

Around 2016, the PROOF TUTOR was incorporated into L&P: the original vision was finally realized.

More important, for this talk, was the expansion of the AProS system to cover set theory for a more advanced logic class, called *Undecidability & Incompleteness*.

Interlude 2.

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More important, for this talk, was the expansion of the AProS system to cover set theory for a more advanced logic class, called *Undecidability & Incompleteness*.

For that class, we also implemented an interface for Gödel proofs. Thus, the perspicuous presentation of Gödel's proofs was brought to pedagogical life.

Stage 3. Gödel Lab – a strategically constructed proof.

Stage 3.

The *Gödel Lab* allows users to give formal proofs of all the theorems mentioned. Let me show you the proof of the underivability of the Gödel sentence G. That proof is a minor variant of the automatic proof I showed you!

Stage 3.

1.	$\vdash(\neg G \leftrightarrow \mathrm{thm}(\ulcorner G \urcorner))$		Prem
2.	(ZFCONS IFF NOT $\vdash(\perp)$)		Prem
3.	ZFCONS	+	Assum
4.	NOT $\vdash(\perp)$		$\leftrightarrow$ER: 2, 3
5.	$\vdash(G)$	+	Assum
6.	$(\neg G \leftrightarrow \mathrm{thm}(\ulcorner G \urcorner))$		ProvE: 1
7.	$\mathrm{thm}(\ulcorner G \urcorner)$		Rep: 5
8.	G		ProvE: 5
9.	$\neg G$		$\leftrightarrow$EL: 6, 7
10.	$\perp$		$\perp$I: 9, 8
11.	$\vdash(\perp)$		ProvI: 10
12.	$\perp\!\!\perp$		$\perp$I: 4, 11
13.	NOT $\vdash(G)$		$\neg$I: 12
14.	(ZFCONS IMPLIES NOT $\vdash(G)$)		$\rightarrow$I: 13

Stage 3.

Gödel contributed to the roundtable discussion with Carnap, Heyting, Waismann, and von Neumann at the Königsberg conference in September of 1930. His contribution is often portrayed as a presentation of the Incompleteness Theorems; here is the actual central assertion:

(Assuming the consistency of classical mathematics) one can even give examples of propositions … that, while contentually true, are unprovable in the formal system of classical mathematics.

Stage 3.

The state of affairs is clarified through Gödel's own account: private discussion with von Neumann after the roundtable session; von Neumann asked, whether the unprovable sentence could be turned into a number theoretic one. Gödel responded to vN's query, that this could be done, but that it would require concepts beyond addition and multiplication.

Then Hao Wang reports:

Shortly afterward Gödel, to his own astonishment, succeeded in turning the undecidable proposition into a polynomial form preceded by quantifiers (over natural numbers). At the same time, but independently of this result, Gödel also discovered his second theorem to the effect that no consistency proof of a reasonably rich system can be formalized in the system itself.

Stage 3.

The argument of the incompleteness theorems is beautifully structured in the 1934 Princeton Lectures.

(i) "indirect definition" of the number theoretic versions of the central meta-mathematical notions. (Gödel discusses "formula" and "proof".)

(ii) "direct definition" of these notions as primitive recursive ones.

(iii) Representability of primitive recursive functions in second order number theory (in Dedekind style).

p. 359: "interesting results"; sections 6 and 7; section 8: "Diophantine equivalents of undecidable prop.".

Postlude.

Postlude.

(**1**) **Space for students**: L&P has been taken since about 2007 by more than 13,000 students. Those students completed the course for full credit at their home institutions. Only a fraction were Carnegie Mellon students.

Empirical investigations: early paper with Scheines; then, later with Patchan, Schunn, and McLaughlin; empirical matters concerning the Gödel Lab were unfortunately interrupted by COVID. –

Effect on non-technical majors!

Postlude.

(**2**) **Methodological framework and natural formalization**: in U&I, systematic development of ZF up to the Cantor-Bernstein Theorem. [10] That involves, in particular, Dedekind's way of making explicit in ZF both e.i.d. and functions defined by structural recursion. That includes, of course, the natural numbers and primitive recursion, thus, the representation of number theory in ZF.

Postlude.

(3) **The presentation of the incompleteness theorems** follows then in U&I quite easily and quickly, as elementary meta-mathematics can be represented in the same way as number theory, thus leading directly to the internalization of syntax! The students prove many interesting results by themselves in the Gödel Lab, for example, Löb's theorem.

Postlude.

Discussing the significance of the incompleteness theorems leads then naturally to Gödel’s reflections on the general notion of “formality”, to computable functions, and in the end to Turing’s analysis of computability. I do consider this material to be a jewel of mathematical, logical, and philosophical reflection.

Thank you!

Thank you for listening, but let me also thank the many, many people who, over more than forty years, have been involved in the various projects.

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