Kurt Gödel and Alfred Tarski: The Extremes of Logic

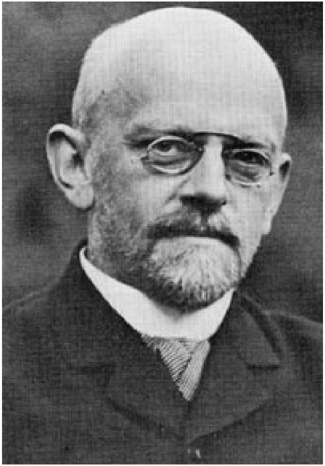
Matthias Baaz

Kurt Gödel

The Gödel Family

1906	April 28	Birth of Kurt Friedrich Gödel, Brünn, Moravia
1916	July 5	Graduation from the evangelische Schule
1916	Fall	Entrance to Staats-Realgymnasium mit deutscher Unterrichtssprache, Brünn
1922	(ca.)	First study of Kant's work
1924	June 19	Graduation from Realgymnasium
1924	Fall 19	Entrance to University of Vienna, with intention seek degree in physics
1925	(ca.)	Adoption of realist position in phylosophy of mathematics
1926		Change of degree focus to mathematics
1926	(ca.)	Beginning of attendance at meetings of Moritz Schlick's circle (the Vienna Circle)
1927	(ca.)	First acquaintance with furute wife, Adele Nimbursky née Porkert)

Adele Gödel



David Hilbert

Real and Ideal Mathematics

Ideal sentences should be eliminable from proofs of real sentences.

Hilbert assumed, that real sentences are decidable.

$$\Downarrow$$

Eliminability of ideal sentences is the same as demonstrating the consistency of the ideal theory.

$$\Downarrow$$

Konsistenz ist Existenz (consistency implies existence).

Luitzen Egbertus Jan Brouwer

Platonism is not sustainable: Why don't we see all truth if we see all the objects?

$$\Downarrow$$

The objects are partial.

$$\Downarrow$$

All transformations are continuous.

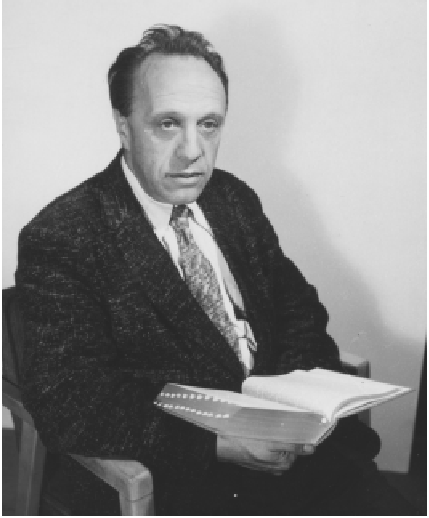
$$\Downarrow$$

Classical logic ($A \vee \neg A$) has to be rejected.

(Gödel developed yet another solution)

1928		Statement of the completeness problem for the first-order predicate calculus in Hilbert and Ackermann 1928; later settled by Gödel in his dissertation
1929		First study of Principia mathematica
1929	February 23	Premature deatch of Rudolf Gödel (father), born on 28 February 1874
1929	June 6	Granting of Austrian citizenship
1929	July 6	Approval of dissertation by Hans Hahn and Philipp Furtwängler
1929	October 22	Submission of revised dissertation to Monatshefte für Mathematik und Physik

Hans Hahn



Karl Menger

Basic Observation

A is not provable from Π

$$\Leftrightarrow$$

$\Pi \cup \{\neg A\}$ is consistent

Construct a model for $\Pi \cup \{\neg A\}$ if $\Pi \cup \{\neg A\}$ is consistent.

Inspired by "Konsistenz ist Existenz"

Corollary (Compactness)

A set of sentences has a model iff any finite subset has a model (only in the publication).

Example

Let $T(\mathbb{R})$ be the set of all sentences true w.r.t. $\mathbb{R}$. Let $S = \{0 < c, 1 < c, 2 < c, \ldots\}$.

$T(\mathbb{R}) \cup S$ is consistent and has a model.

$\Rightarrow \frac{1}{c}$ is an infinitesimal

$\Rightarrow$ non-standard analysis exists.

1929	October 24	First regular meeting of Karl Menger's mathematics colloquium, and Gödel's first attendance there (at Menger's invitation); thirteen contributions by Gödel published in the colloquium proceedings during 1932 – 1936; volumes 1 – 5, 7 and 8 edited with the assistance of Gödel (and others)
1930	August 26	Private announcement to Carnap, Feigl and Waismann of the (first) incompleteness results, during discussion at the Café Reichsrat
1930	September 7	First public announcement of the existence of formally undecidable propositions in number theory, during discussion session at Königsberg
1930	November 17	Receipt of the incompleteness paper by Monatshefte für Mathematik und Physik
1930	November 20	Letter to Gödel from von Neumann, announcing his own independent discovery of the unprovability of consistency

Über Formal Unentscheidbare Sätze der Principia Mathematica und Verwandter Systeme 1

$D1 \quad T \vdash \varphi \quad \text{implies} \quad S \vdash PR_T(\ulcorner \varphi \urcorner)$

$D2 \quad S \vdash PR_T(\ulcorner \varphi \urcorner) \rightarrow PR_T(\ulcorner PR_T(\ulcorner \varphi \urcorner) \urcorner)$

$D3 \quad S \vdash PR_T(\ulcorner \varphi \urcorner) \wedge PR_T(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow PR_T(\ulcorner \psi \urcorner)$

Theorem (Diagonalization Lemma)

Let φx in the language of T have only the free variable indicated. Then there is a sentence ψ such that

$$S \vdash \psi \leftrightarrow \varphi(\ulcorner \psi \urcorner).$$

Proof

Given φx, let $\theta x \leftrightarrow \varphi(sub(x,x))$ be the diagonalization of φ. Let $m = \ulcorner \theta x \urcorner$ and $\psi = \theta m$. Then we claim

$$S \vdash \psi \leftrightarrow \varphi(\ulcorner \psi \urcorner).$$

For, in S, we see that

$$\psi \leftrightarrow \theta m \leftrightarrow \varphi(sub(m,m)) \leftrightarrow \varphi(sub(\ulcorner \theta m \urcorner, m))(\text{ since } m = \ulcorner \theta x \urcorner)$$

$$\leftrightarrow \varphi(\ulcorner \theta m \urcorner) \leftrightarrow \varphi(\ulcorner \psi \urcorner)$$

We apply this to $\neg PR_T(x)$.

Theorem (First Incompleteness Theorem)

Let $T \vdash \varphi \leftrightarrow \neg PR_T(\ulcorner \varphi \urcorner)$. Then

1. $T \nvdash \varphi$;
2. under an additional assumption, $T \nvdash \neg\varphi$.

Proof

1. Observe that $T \vdash \varphi$ implies $T \vdash PR_T(\ulcorner \varphi \urcorner)$, by $D1$, which implies $T \vdash \neg\varphi$, contradicting the consistency of T.
2. The additional assumption is a strengthening of the converse to $D1$, namely $T \vdash PR_T(\ulcorner \varphi \urcorner)$ implies $T \vdash \varphi$.

We have $T \vdash \neg\varphi$, hence $T \vdash \neg\neg PR_T(\ulcorner \varphi \urcorner)$ so that $T \vdash PR_T(\ulcorner \varphi \urcorner)$ and, by the additional assumption, $T \vdash \varphi$, again contradicting the consistency of T.

Theorem (Second Incompleteness Theorem, ein merkwürdiger Satz)

Let Con_T be $\neg PR_T(\ulcorner \Lambda \urcorner)$, where Λ is any convenient contradictory statement. Then $T \nvdash Con_T$.

Proof

Let φ be as before. We show $S \vdash \varphi \leftrightarrow Con_T$.
Observe that $S \vdash \varphi \rightarrow \neg PR_T(\ulcorner \varphi \urcorner)$ implies $S \vdash \varphi \rightarrow \neg PR_T(\ulcorner \Lambda \urcorner)$, since $T \vdash \Lambda \rightarrow \varphi$ implies $S \vdash PR_T(\ulcorner \Lambda \rightarrow \varphi \urcorner)$, by $D1$, which implies $S \vdash PR_T(\ulcorner \Lambda \urcorner) \rightarrow PR_T(\ulcorner \varphi \urcorner)$, by $D3$.
But $\varphi \rightarrow \neg PR_T(\ulcorner \Lambda \urcorner)$ is just $\varphi \rightarrow Con_T$ and we have proven half of the equivalence.
Consequently, by $D2$, $S \vdash PR_T(\ulcorner \varphi \urcorner) \rightarrow PR_T(\ulcorner PR_T(\ulcorner \varphi \urcorner) \urcorner)$, which implies $S \vdash PR_T(\ulcorner \varphi \urcorner) \rightarrow PR_T(\ulcorner \neg \varphi \urcorner)$, by $D1$, $D3$, since $\varphi \leftrightarrow \neg PR_T(\ulcorner \varphi \urcorner)$. This yields $S \vdash PR_T(\ulcorner \varphi \urcorner) \rightarrow PR_T(\ulcorner \varphi \wedge \neg \varphi \urcorner)$, by $D1$, $D3$ and logic. By contraposition, $S \vdash \neg PR_T(\ulcorner \Lambda \urcorner) \rightarrow \neg PR_T(\ulcorner \varphi \urcorner)$, which is $S \vdash Con_T \rightarrow \varphi$, by definitions.

Zur Intuitionistic Arithmetic und Zahlentheorie (1933)
(on intuitionistic arithmetic and number theory)

$$\neg p \Rightarrow \neg p$$

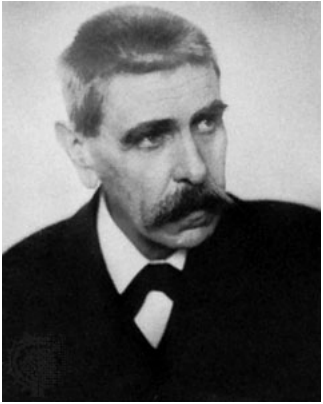
$$p \rightarrow q \Rightarrow \neg(p \wedge \neg q)$$

$$p \vee q \Rightarrow \neg(\neg p \wedge \neg q)$$

$$p \wedge q \Rightarrow p \wedge q$$

$$\exists y P(x) \Rightarrow \neg \forall x \neg P(x)$$

1931	September 15	Presentation of the incompleteness results to the annual meeting of the Deutsche Mathematiker-Vereinigung, Bad Elster; meeting and discussion with Zermelo
1932	June 25	Submission of 1931 to the University of Vienna as Gödels Habilitationsschrift
1932	December 1	Granting of Habilitation
1933	March 11	Granting of Dozentur; Gödel's first course (foundations of arithmetic) taught shortly thereafter
1934	February-March	I.A.S. lectures on the incompleteness results
1934	July 24	Death of Hans Hahn
1934	Fall	Admission to senatorium in Purkersdorf bei Wien for treatment of schizophrenia



Julius Wagner von Jauregg

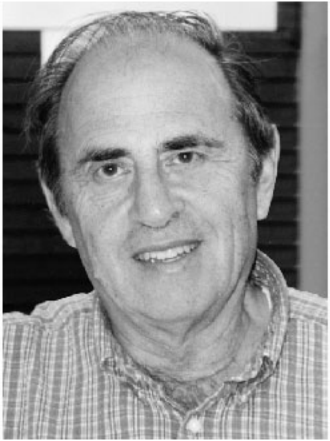
1936	Winter-spring	More time in sanatoria
1936	June 22	Schlick murdered by a deranged ex-student
1937	June 14	Discovery of crucial step in proof of relative consistency of the generalized continuum hypothesis
1938	March 13	Hitler's annexation of Austria (Anschluss)
1938	September 20	Merriage in Vienna to Adele Nimbursky
1938	November 9	Set-theoretic consistency results communicated to the National Academy of Sciences
1940	January 8	Issuance of U.S. non-quota immigrant visas
1940	January 12	Issuance in Berlin of Russian transit visas
1940	January 18 to March 4	Journey to America via trans-Siberian railway and U.S ship President Cleveland (Yokohama to San Francisco)

The consistency of the axiom of choice and of the generalized continuum hypothesis

The second incompleteness theorem implies that it is hard to prove the consistency of ZF + AC + CH, a very strong theory. However do we mind the consistency of ZF + AC + CH if ZF is not consistent?

$$\Downarrow$$

We assume the consistency of ZF and obtain by the completeness theorem a model. We change this mode to a model of ZF + AC + CH.



Paul Cohen

Paul Cohen: the independence of the continuum hypothesis (1962)

Gödel: What is the idea of the proof?

Cohen explains the proof

Gödel: What is the idea of the proof?

and so on

The younger generation scholars of Kurt Gödel

Georg Kreisel

Gaisi Takeuti

Hao Wang

1944	July	First publication of results in relativity theory
1958		Appearance of 1958 in Dialectica (last published paper apart from revisions of earlier works)
1976	July 1	Retirement from I.A.S., as professor emeritus
1977	July	Hospitalization of Adele Gödel for major surgery
1977	July 26	Death, in Princeton, of Oskar Morgenstern, Gödels closest personal friend
1977	Mid December	Release of Adele from hospital
1977	December 29	Hospitalization of Gödel himself, ad Adele's urging
1978	January 14	Death of Kurt Gödel, at Princeton Hospital, due to "malnutrition and inanition"

Kurt Gödel and Albert Einstein

A universe where timelines may loop (the rotating universe)

Einstein/Gödel

Über eine noch nicht benützte Erweiterung des finiten Standpunktes

Let

$$F' = (\exists y)(z)A(y, z, x)$$

$$G' = (\exists v)(w)B(v, w, u)$$

be already defined; then we have by definition

1. $(F' \wedge G)' = (\exists yv)(zw)[A(y, z, x) \wedge B(v, q, u)]$
2. $(F' \vee G)' = (\exists yvt)(zw)[t = 0 \wedge A(y, z, x). \vee .t = 1 \wedge B(v, q, u)]$
3. $[(s)F]' = (\exists Y)(sz)A(Y(s), z, x)$
4. $[(\exists s)F]' = (\exists sy)(z)A(y, z, x)$
5. $(F \supset G)' = (\exists VZ)(yw)[A(y, Z/yw), x) \supset B(V(y), w, u)]$
6. $(\neg F)' = (\exists \overline{Z})(y)\neg A(y, \overline{Z}(y), x)$

Trick: $\forall x \exists y A(x, y) \Rightarrow \exists X(x) \forall x A(x, X(x))$

Thinking is unlimited

Mathematical intuition is the key for sustaining platonism

New axioms have to be detected

David Hilbert: “All problems are solvable”

Thinking is effective

Or the existence of two Eulers


Leonhard Euler

Thoughts connecting distant fields of science are effective proportional to the distance

Ontological proof $G(x) =_{Df}$ x

$NE(x) \equiv_{Df} N(\exists y)Ess_x(y)$	necessary existence
$G(x) \supset NE(x)$	since NE is a positive property
$G(x) \supset .Ess_x \supset G^o$	holds for any property in place of G
$G(x) \supset N(\exists y)G(y)$	follows from the 3 preceding
$(\exists x)G(x) \supset N(\exists y)G(y)$	
$M(\exists x)G(x) \supset MN(\exists y)G(y)$	addition of M on both sides
hence $\supset N(\exists y)G(y)$	

Kurt Gödel, the most important logician since Aristotle

(or is it better to compare Gödel to Archimedes?)

An Epitaph for Gödel

Rudolf Gödel to Georg Kreisel

"Among us: Has my brother done anything reasonable in his life?"

Georg Kreisel to Rudolf Gödel

"He has proven a small theorem, similar to the prime-number theorem of Euclid."

Alfred Tarski

1901	Jan 14	Born in Warsaw as Alfred Teitelbaum
1923		Conversion to Roman Catolizism, change of name to Tarski
1928		Married to Maria Witkowska (son and daughter)
1930		Visit of Vienna
		Lecture in Karl Menger's colloquium
1935		Fellowship in Vienna
1939		Unity of Science Congress Harvard
1942		University of California at Berkeley
1983	Oct 26	Died in Berkeley

The Non-Definability of Truth

"It snows" is true iff it snows

minimal condition

$T(\ulcorner A \urcorner) \leftrightarrow A$

but there is a sentence L with

$\neg T(\ulcorner L \urcorner) \leftrightarrow L$

contradiction

Tarski on Logical Consequence

$$\Gamma \models A$$

if Γ is finite this reflects mainly into logic (cf for classical propositional logic)

$$\models \bigwedge_{B \in \Gamma} B \supset A$$

"preservation of truth"

The Banach-Tarski Paradox

A sphere can be dissected into finitely many parts and from this parts two such spheres can be reassembled.

Quantifier Elimination

Consider the dense order $<$ on $[0,1]$

$$\exists x A(x,\bar{y}) \leftrightarrow \exists x (\bigvee_i \bigwedge_j B_{ij}(x,\bar{y}))$$

where

$$B_{ij}(x,\bar{y}) = \begin{cases} s = t & OR \\ \neg s = t & OR \\ s < t & OR \\ \neg s < t & \end{cases}$$

Replace $\neg s = t$ by $s < t \vee t < s$ and $\neg s < t$ by $t < s \vee t = s$.
Replace $x = x$ by T, $x < x$ by F. Confine $\exists$ to x.
By dichotomy we obtain the following cases

$$\exists x(s = x \wedge A(x)) \leftrightarrow A(s)$$

otherwise

$$\exists x(s < x \wedge x < t) \leftrightarrow s < t$$

$$\exists x(s < x) \leftrightarrow s < 1$$

$$\exists x(x < t) \leftrightarrow 0 < t$$

Tarski's Theorem

RCF

- the axioms of ordered fields
- the axiom asserting that every positive number has a square-root
- for every odd n the axiom asserting that all polynomials of degree n have a root

Tarski's theorem states that RCF is complete (hence decidable= and admits quantifier elimination

Students (Not Complete)

- Evert Willem Beth
- Chen Chung Chang
- Solomon Feferman
- Haim Gaifman
- Bjarn Jonsson
- Howard Jerome Keisler
- Roger Maddox
- Richard Montague
- Anne C Morel
- Andrzej Mostowski
- Julia Robinson
- Wanda Szmielew
- Robert Vaught

Julia Robinson

Definability of $\mathbb{Z}$ by $\mathbb{Q}$

$$A(r, y, z) \equiv \exists a \exists b \exists c (2 + y * z * r^2 = a^2 + y * b^2 + x * c^2)$$

$$B(r, y, z) \equiv (A(0, y, z) \wedge \forall w((A(w, y, z) \supset A(w+1, y, z)) \supset A(w, y, z))$$

$$\Downarrow$$

$$r \in \mathbb{Z} \Leftrightarrow \forall x \forall z B(r, y, z)$$

$$(r \in \mathbb{N} \Leftrightarrow \exists a \exists b \exists c \exists d (r = a^2 + b^2 + c^2 + d^2) \text{ on } \mathbb{Z})$$

Richard Montague

Compositional higher order semantics for natural languages

Evert Willem Beth

First order logic implicitly definable

$$\Sigma(P) \vee \Sigma(P') \vdash \forall x(P(x) \leftrightarrow P'(x))$$

explicitly definable

$$\Sigma(P) \vdash \forall x(P(x) \leftrightarrow I(x)),$$

I not containing P

Beth's theorem

$$\text{implicitly definable} \Leftrightarrow \text{explicitly definable}$$

Tractatus Logico-Philosophicus, Wittgenstein

5.511

How can the all-embracing logic which mirrors the world use such special catches and manipulations? Only because all these are connected into an infinitely fine network, to the great mirror.