Equivalence relations and Borel reduction

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Outline

1. Equivalence relations and invariants
2. Dichotomy theorems
3. Orbit equivalence relations
4. Σ^1_1 equivalence relations

Classical examples (1)

Example

For $A, B \in \mathbb{C}^{m\times n}$, define $A \sim B \iff A = TBS$, where T, S are invertible matrices.

Let $r(A)$ be rank of A. Then

$$A \sim B \iff r(A) = r(B).$$

Example

For $A, B \in \mathbb{C}^{n\times n}$, define $A \approx B \iff A = TBT^{-1}$, where T is an invertible matrix.

Let $J(A)$ be the Jordan normal form of A. Then

$$A \approx B \iff J(A) = J(B).$$

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Classical examples (2)

Example

Every finitely generated abelian group G is isomorphic to a direct sum

$$\mathbb{Z}^m \oplus \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{e_i} \mathbb{Z}(p_i^j)^{t_{ij}}.$$

Let $M(G) = (m, t_{ij})_{i \leq n, j \leq e_i}$. Then

$$G \cong H \iff M(G) = M(H).$$

Note: $r(A), J(A), M(G)$ are not continuous mappings.

Classical examples (3)

Example

For compact topological spaces X, denote $C(X)$ the spaces of all continuous function $X \to \mathbb{C}$ equipped with the sup norm.

Theorem (Gelfand-Naimark)

Let X, Y be compact Hausdorff spaces. Then X is homeomorphic to Y iff $C(X)$ is isomorphic to $C(Y)$ (as a C-algebra).*

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Reduction

Definition

Let E, F be two equivalence relations on X, Y respectively, $\theta : X \to Y$ is a **reduction** of E to F if

$$aEb \iff \theta(a)F\theta(b)$$

for $a, b \in X$.

Fact

Let $f : X/E \to X$ be a choice function, and let $\theta(a) = f([a]_E)$. Then θ is a reduction of E to $\mathrm{id}(X)$.

Note: We need some restrictions on reduction mapping!

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Polish spaces

Definition

Polish space: a separable, completely metrizable topological space.

Example

1. $\mathbb{C}^{m\times n}$; $\mathbb{R}^n$; separable Banach spaces;
2. $[0,1]$; $(0,1)$;
3. Cantor space $\{0,1\}^{\mathbb{N}}$; Baire space $\mathbb{N}^{\mathbb{N}}$;
4. For a countable abelian group $(G,\oplus)$, note that

$$R_G = \{(a,b,c) : a = b \oplus c\} \subseteq \mathbb{N}^3.$$

"finitely generated abelian groups" is a Borel subset of $\{0,1\}^{\mathbb{N}^3} \cong \{0,1\}^{\mathbb{N}}$.

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Borel sets, Borel functions and Borel reductions

Definition

$\mathbf{B}(X)$: **Borel sets** of X is the σ-algebra generated by open sets.

$\mathbf{B}(X)$ contains all open, closed, $F_\sigma, G_\delta, G_{\delta\sigma}, F_{\sigma\delta}$, etc., sets.

Let X, Y be two Polish spaces.

Definition

A function $f : X \to Y$ is **Borel function** if $f^{-1}(U)$ is Borel for U open in Y.

Let E, F be equivalence relations on X, Y respectively.
$E \leq_B F$: There is a Borel reduction of E to F;
$E \sim_B F$: $E \leq_B F$ and $F \leq_B E$;
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Standard Borel spaces

Definition

A measurable space $(X, \mathcal{S})$ is a *standard Borel space* if there is a Polish topology τ on X with $\mathcal{S} = \mathbf{B}(X, \tau)$.

Theorem

Let X be a Polish space, $Y \subseteq X$. Then $(Y, \mathbf{B}(Y))$ is a standard Borel space iff Y is a Borel subset of X.

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"finitely generated abelian groups" is a standard Borel space.

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Effros Borel spaces

Example (Effros Borel spaces)

Given a Polish space X, we denote by $F(X)$ the set of closed subsets of X. We endow $F(X)$ with the σ-algebra generated by the sets

$$\{F \in F(X) : F \cap U \neq \emptyset\},$$

where U varies over open subsets of X.

Fact

If X is Polish, the Effros Borel space of $F(X)$ is a standard Borel space.

About Gelfand-Naimark's theorem

Fact

1. *Every compact metric space homeomorphic to a closed subset of Hilbert Cub* $[0,1]^{\mathbb{N}}$.
2. *Every separable Banach space isometrically isomorphic to a closed linear subspace of* $C[0,1]$.

Let $\mathrm{Hom}_{\mathrm{cpt}}$ be the homeomorphism relation on $F([0,1]^{\mathbb{N}})$, and let $\cong_{\mathrm{SB}}$ be the isometrically isomorphism on $\mathrm{Subs}(C[0,1]) \subseteq F(C[0,1])$ of all closed linear subspaces of $C[0,1]$. Then

$$\mathrm{Hom}_{\mathrm{cpt}} \leq_B \cong_{\mathrm{SB}} .$$

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Smooth equivalence relations

We denote by $\mathrm{id}(X)$ the *identity relation* on X.

$$\mathrm{id}(n) <_B \mathrm{id}(\mathbb{N}) <_B \mathrm{id}(\mathbb{R}).$$

Definition

We say E is **smooth** if $E \leq_B \mathrm{id}(\mathbb{R})$.

Fact

Let X, Y be Polish spaces, then X is Borel isomorphic to Y (i.e., there is a Borel bijection from X to Y) iff $|X| = |Y|$. So

$$\mathrm{id}(\mathbb{R}) \sim_B \mathrm{id}(\mathbb{R}^n) \sim_B \mathrm{id}(\mathbb{R}^{\mathbb{N}}) \sim_B \mathrm{id}(\{0,1\}^{\mathbb{N}}).$$

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$\mathbf{\Sigma}^1_1$ sets and $\mathbf{\Pi}^1_1$ sets

Definition

Let X be a Polish space. A subset $A \subseteq X$ is **analytic** (or $\mathbf{\Sigma}^1_1$) if there is a Polish space Y and a closed subset $C \subseteq X \times Y$ such that

$$x \in A \iff \exists y \in Y((x, y) \in C).$$

A subset $A \subseteq X$ is **co-analytic** (or $\mathbf{\Pi}^1_1$) if $X \setminus A$ is $\mathbf{\Sigma}^1_1$.

Theorem (Suslin)

Let $A \subseteq X$. Then A is Borel iff it is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

fact: All $\sigma(\mathbf{\Sigma}^1_1)$ sets in $\mathbb{R}$ are Lebesgue measurable.

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Non-smooth equivalence relations

T is a **transversal** for E if T meets each E-class at exactly one point.

Theorem

Let E be an equivalence relation on a Polish space X. If E is smooth, then E has a $\sigma(\mathbf{\Sigma}^1_1)$ transversal.

Definition

Vitali equivalence relation: For $x, y \in \mathbb{R}$ we define

$$xE_vy \iff x - y \in \mathbb{Q}.$$

Note: Any transversal of E_v, i.e., any vitali set, is not Lebesgue measurable, so E_v is not smooth.

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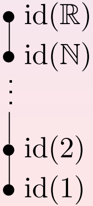
1st dichotomy theorem

We say an equivalence relation E on X is Borel, $\boldsymbol{\Sigma}^1_1$, or $\boldsymbol{\Pi}^1_1$ if $\{(x,y)\in X^2 : xEy\}$ is so in X^2.

Theorem (Silver, 1980)

Let E be a $\boldsymbol{\Pi}^1_1$ equivalence relation. Then

$$E \leq_B \mathrm{id}(\mathbb{N}) \text{ or } \mathrm{id}(\mathbb{R}) \leq_B E.$$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2nd dichotomy theorem

Definition

E_0 is the equivalence relation on $\{0,1\}^{\mathbb{N}}$ defined by

$$xE_0y \iff \exists m \forall n \geq m(x(n) = y(n)).$$

Fact: $E_0 \sim_B E_v = \mathbb{R}/\mathbb{Q}$.

Theorem (Harrington-Kechris-Louveau, 1990)

Let E be a Borel equivalence relation. Then either $E \leq_B \mathrm{id}(\mathbb{R})$ or $E_0 \leq_B E$.

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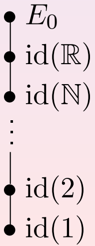
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3rd dichotomy theorem

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E_1 is the equivalence relation on $\mathbb{R}^{\mathbb{N}}$ defined by

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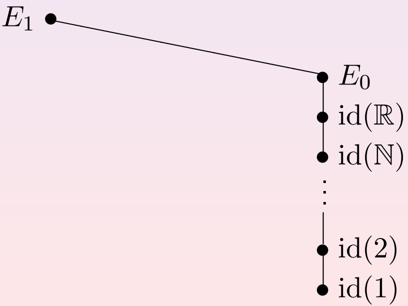
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4th dichotomy theorem

Definition

Let E be an equivalence relation on X. The equivalence relation E^ω on $X^\mathbb{N}$ defined by

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Fact: $E_0^\omega \sim_B E_v^\omega = \mathbb{R}^\mathbb{N}/\mathbb{Q}^\mathbb{N}$.

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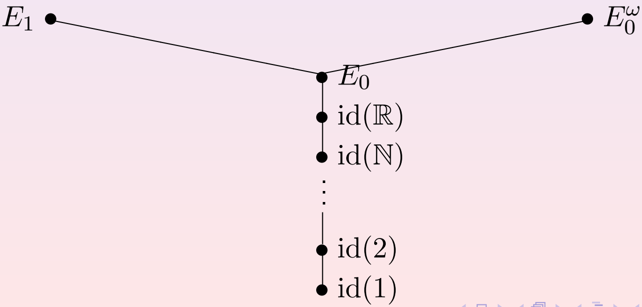
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Polish G-spaces and orbit equivalence relations

Definition

Polish group: A topological group whose underlying space is Polish.

G: Polish group,
X: Polish space,
$a : G \times X \to X$: continuous G-action on X.

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Orbit equivalence relation:

$$x E_G^X y \iff \exists g \in G (g \cdot x = y).$$

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Polish groups

1. Countable groups with discrete topology.
2. S_∞: all bijections of $\mathbb{N} \to \mathbb{N}$ with pointwise convergence topology.
3. Separable Banach spaces with addition.
4. **Universal Polish group** $H([0,1]^{\mathbb{N}})$: every Polish group is isomorphic to one of its closed subgroups. (Uspenskii, 1986)
5. **Surjectively universal Polish group** $\overline{F}_\Gamma(\mathcal{N}_\omega)$: every Polish group is isomorphic to one of its topological quotient group. (D. 2012)

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Countable Borel equivalence relations

An equivalence relation E on X is countable if every equivalence class of E is countable.

Theorem (Feldman – Moore)

Let E be a countable Borel equivalence relation on a Polish space X. Then $E = E_G^X$ for some countable discrete group G.

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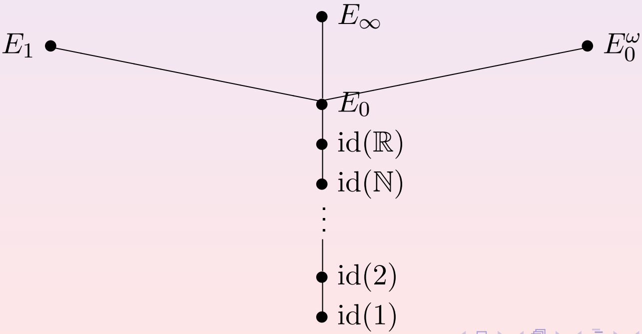
Theorem (Feldman – Moore)

Let E be a countable Borel equivalence relation on a Polish space X. Then $E = E_G^X$ for some countable discrete group G.

E_∞: an universal countable Borel equivalence relation.


E_∞
E_1
E_0^ω
E_0
id($\mathbb{R}$)
id($\mathbb{N}$)
id(2)
id(1)

Below E_0

Theorem (Dougherty-Jackson-Kechris, 1994)

$E_{\mathbb{Z}}^X \leq_B E_0$.

Theorem (Jackson-Kechris-Louveau, 2002)

$E_{\mathbb{R}^n}^X \leq_B E_0$.

Theorem (Gao-Jackson, 2015)

For any countable abelian discrete group G, $E_G^X \leq_B E_0$.

Theorem (D.-Gao, 2017)

Let G be a abelian closed subgroup of S_∞ and $E_G^X \leq_B E_\infty$, then $E_G^X \leq_B E_0$.

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S_∞-spaces

Borel complete $E^\infty_{S_\infty}$: every $E^X_{S_\infty}$ is Borel reducible to it.

Theorem

The isomorphism relations of all countable graphs, countable trees, countable linear orderings and countable groups are Borel complete, i.e.,

$$E^\infty_{S_\infty} \sim_B (\cong_{\text{Graph}}) \sim_B (\cong_{\text{Tr}}) \sim_B (\cong_{\text{LO}}) \sim_B (\cong_{\text{Group}}).$$

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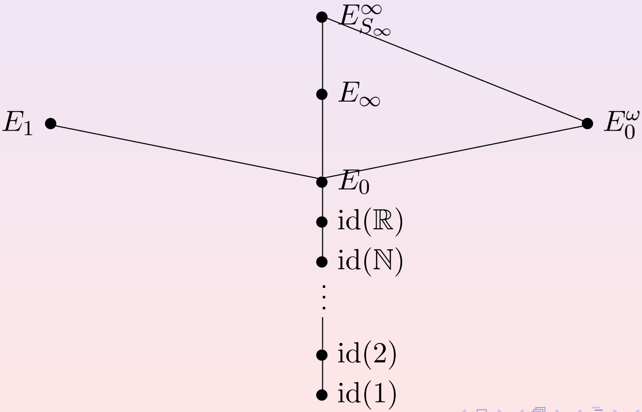
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Upper bound of E_G^X

Theorem (Gao-Kechris, 2003)

Let $\cong_{\rm PM}$ be the isometric isomorphism relation between Polish metric spaces. Then $E_G^X \leq_B \cong_{\rm PM}$ for every Polish G-space X.

Theorem (Melleray-Weaver, 2007)

$\cong_{\rm PM} \sim_B \cong_{\rm SB}$.

Theorem (Zielinski, 2015)

$\cong_{\rm SB} \sim_B \ {\rm Hom}_{\rm cpt}$.

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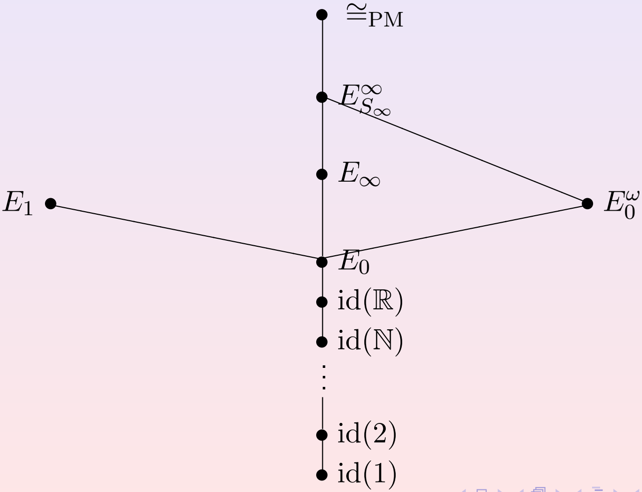
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Outline

1. Equivalence relations and invariants
2. Dichotomy theorems
3. Orbit equivalence relations
4. Σ_1^1 equivalence relations

Σ^1_1 equivalence relations

Theorem (Kechris-Louveau, 1997)

$E_1 \not\leq_B E_G^X$ *for any Polish* G*-space* X.

Upper bound of Σ^1_1 equivalence relations

1. $\cong^{\rm L}_{\rm SB}$: separable Banach spaces, linear isomorphism;
2. $\mathrm{Hom}^{\rm Lip}_{\rm SB}$: separable Banach spaces, Lipschitz isomorphism;
3. $\mathrm{Hom}^{\rm U}_{\rm PM}$: Polish metric spaces, uniform homeomorphism ;
4. $\cong_{\rm AG}$: abelian Polish groups, topological isomorphism.

Theorem (Ferenczi-Louveau-Rosendal, 2009)

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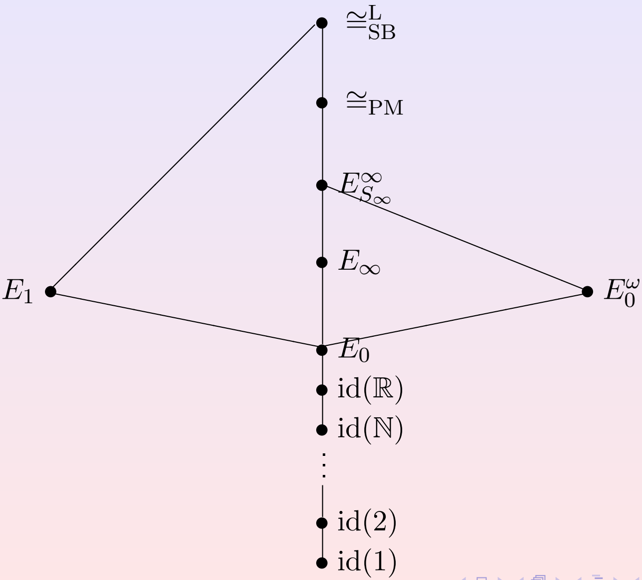
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The end

Thank you!