Kripke-Platek set theory

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Σ_1-formulas in arithmetic

Consider the standard model of first-order arithmetic $(\mathbb{N}, \leqslant, 0, 1, +, \times)$, we will denote it simply as $\mathbb{N}$.

First-order arithmetical formula φ is a Δ_0-formula if all quantifiers in φ are bounded, i.e. all occurences of quantifiers are of the form

$$(\forall x \leqslant t)\psi(x) \equiv \forall x(x \leqslant t \to \psi(x)) \text{ or } (\exists x \leqslant t)\psi(x) \equiv \exists x(x \leqslant t \wedge \psi(x)),$$

where $x \notin FV(\psi)$.

Examples:
Formulas $x = x$, $\exists y \leqslant x(y + y = x)$, $(\forall y \leqslant x)(\forall z \leqslant y)(yz = x \to z = 1)$ are Δ_0.
Formulas $\exists y(y + y = x)$, $(\forall x \leqslant x + x)x = 0$ aren't Δ_0.

The class Σ_1 consists of all formulas of the form $\exists x_1, \ldots, x_n\, \varphi(x_1, \ldots, x_n)$, where $\varphi \in \Delta_0$.

Computability and Σ_1-definability

A set $A \subseteq \mathbb{N}^k$ is defined by $\varphi(x_1, \ldots, x_k)$ if for any $n_1, \ldots, n_k \in \mathbb{N}$

$$\langle n_1, \ldots, n_k \rangle \in A \iff \mathbb{N} \models \varphi(n_1, \ldots, n_k).$$

A function $f\colon \mathbb{N}^k \to \mathbb{N}$ is defined by $\varphi(x_1, \ldots, x_k, y)$ if for any $n_1, \ldots, n_k, m \in \mathbb{N}$

$$f(n_1, \ldots, n_k) = m \iff \mathbb{N} \models \varphi(n_1, \ldots, n_k, m).$$

A partial function $f\colon A \to \mathbb{N}$, $A \subseteq \mathbb{N}^k$ is defined by $\varphi(x_1, \ldots, x_k, y)$ if for any $n_1, \ldots, n_k, m \in \mathbb{N}$

$$\langle n_1, \ldots, n_k \rangle \in A \;\&\; f(n_1, \ldots, n_k) = m \iff \mathbb{N} \models \varphi(n_1, \ldots, n_k, m).$$

Proposition

A set $A \subseteq \mathbb{N}^k$ is computably enumerable iff it is Σ_1-definable in $\mathbb{N}$. A set $A \subseteq \mathbb{N}^k$ is computable iff both it and its complement are Σ_1-definable in $\mathbb{N}$. A function f (partial function f) on $\mathbb{N}$ is computable iff it is Σ_1-definable in $\mathbb{N}$.

Computability in $\mathbb{HF}$

The set of all hereditary finite sets

$$\mathbb{HF} = \varnothing \cup \mathcal{P}(\varnothing) \cup \mathcal{P}(\mathcal{P}(\varnothing)) \cup \ldots \cup \mathcal{P}^n(\varnothing) \cup \ldots$$

We identify $\mathbb{HF}$ with the structure $(\mathbb{HF}, \in)$.

A set-theoretic formula φ is Δ_0 if all quantifiers in φ are bounded, i.e. all occurrences of quantifiers are of the form

$$(\forall x \in y)\psi(x) \equiv \forall x(x \in y \to \psi(x)) \text{ or } (\exists x \in y)\psi(x) \equiv \exists x(x \in y \wedge \psi(x)),$$

where x and y are distinct variables.

Proposition

A set $A \subseteq \mathbb{HF}^k$ is computably enumerable iff it is Σ_1-definable in $\mathbb{HF}$. A set $A \subseteq \mathbb{HF}$ is computable iff both it and its complement are Σ_1-definable in $\mathbb{HF}$. A function f (partial function f) on $\mathbb{HF}$ is computable iff it is Σ_1-definable in $\mathbb{HF}$.

Computability over a structure

Suppose $\mathfrak{M} = (M, R_1, \ldots, R_n)$ is some first-order structure and $R_1, \ldots, R_n$ are relations on it.
The set $\mathbb{HF}(\mathfrak{M})$ is the set of all hereditary finite sets with urelements from $\mathfrak{M}$

$$\mathbb{HF}(\mathfrak{M}) = M \cup \mathcal{P}^{<\omega}(M) \cup \mathcal{P}^{<\omega}(M \cup \mathcal{P}^{<\omega}(M)) \cup \ldots$$

Here elements of M considered to be urelements, i.e. objects that could not have elements.

Σ_1-definability in $(\mathbb{HF}(\mathfrak{M}), R_1, \ldots, R_n)$ corresponds to computations that could manipulate with finite collections of objects from $\mathfrak{M}$ and could use predicates $R_1, \ldots, R_n$.

Constructible sets

- $L_0 = \varnothing$
- $L_{\alpha+1} = \text{Def}(L_\alpha) = \{a \subseteq L_\alpha \mid$
  a is definable with parameters in $(L_\alpha, \in)\}$
- $L_\lambda = \bigcup_{\beta<\lambda} L_\beta$, for limit ordinals λ

Note that $\mathbb{HF} = L_\omega$.

Computations in L_α?

We would like to equate computability relativized to L_α with Σ_1-definability.

However for most α's we don't get decent computability notion.

1. If α isn't limit, then Cartesian product $\times$ isn't a total function.
2. If α isn't of the form ω^β, then the ordinal addition isn't total.
3. In $L_{\omega+1}$ there is a set $a \in L_{\omega+1}$ and Σ_1-definable f, $\mathrm{dom}(f) \supseteq a$ such that $\{f(x) \mid x \in a\} \notin L_{\omega+1}$. Namely we have Σ_1-definable function $TJ\colon n \in \omega \longmapsto \varnothing^{(n)}$, where $\varnothing^{(n)} \subseteq \omega$ is n-th Turing jump of $\varnothing$. However, the image $\{TJ(n) \mid n \in \omega\} \notin L_{\omega+1}$.
4. Analogue of 3. holds in any L_α, where α is computable, i.e. $\alpha = \mathrm{ot}(\prec)$, for some computable well-ordering $\prec$ of $\mathbb{N}$.

In fact the first decent setting for computability after L_ω is provided by $L_{\omega_1^{CK}}$, where

$$\omega_1^{\mathrm{CK}} = \sup\{\mathrm{ot}(\prec) \mid \prec \text{ is a computable well-ordering of } \mathbb{N}\}$$

Kripke-Platek set theory KP

KP is a first-order theory in the language of set theory

Axioms of KP

1. $\forall z\, (z \in x \leftrightarrow z \in y) \to x = y$ (Extensionality)
2. $\exists z(x \in z \wedge y \in z)$ (Pair)
3. $\exists y(\forall z \in x)(\forall w \in z)w \in y$ (Union)
4. $\exists y \forall z(z \in y \leftrightarrow z \in x \wedge \varphi(z))$, where φ is Δ_0 and $y \notin FV(\varphi)$ (Δ_0-Separation)
5. $(\forall y \in x)\exists z\, \varphi(x, y, z) \to \exists z_0(\forall y \in x)(\exists z \in z_0)\varphi(x, y, z)$, where φ is Σ_1 and $z_0 \notin FV(\varphi)$ (Σ_1-collection)
6. $\exists x\, \varphi(x) \to \exists x\, (\varphi(x) \wedge (\forall y \in x)\neg\varphi(x))$ (Foundation)

Set theories with urelements

Variables x, y, z, w range over any objects.
Variables a, b, c, d range over sets.
Variables p, q, r range over urelements.

We have a predicate $\in$ and some relations $R_1, \dots, R_n$ on urelements. We presuppose that

1. any object is either set or urelment but not both;
2. urelements do not have elements
3. $R_i(x_1, \dots, x_{k_i})$ should be false if at least one of x_j isn't an urelement

Δ_0-formulas are formulas, where all quantifiers are bounded, e.g. $(\forall x \in a)$, $(\exists p \in x)$.
Σ_1-formulas of the form $\exists \vec{x} \exists \vec{a} \exists \vec{p} \varphi$, where φ is Δ_0.

Formally we could simulate this in a one-sorted first-order language with a predicate $\mathrm{Ur}(x)$,

Theory KPU

Axioms of KPU

1. $\forall z\ (z \in a \leftrightarrow z \in b) \rightarrow a = b$ (Extensionality)
2. $\exists a(x \in a \wedge y \in a)$ (Pair)
3. $\exists b(\forall c \in a)(\forall x \in c)x \in b$ (Union)
4. $\exists b \forall x(x \in b \leftrightarrow x \in b \wedge \varphi(x))$, where φ is Δ_0 and $b \notin FV(\varphi)$ (Δ_0-Separation)
5. $(\forall x \in a)\exists y\ \varphi(a, x, y) \rightarrow \exists b(\forall x \in a)(\exists y \in b)\varphi(a, x, y)$, where φ is Σ_1 and $b \notin FV(\varphi)$ (Σ_1-collection)
6. $\exists x\ \varphi(x) \rightarrow \exists x\ (\varphi(x) \wedge (\forall y \in x)\neg\varphi(x))$ (Foundation)

Cartesian product

Kuratowski pairing: $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$.
Cartesian product: $a \times b = \{\langle x, y\rangle \mid x \in a \text{ and } y \in b\}$

Theorem

Existence of Cartesian product is provable in KPU.

Proof.

We prove that $a \times b$ exists.
The following Δ_0 formula expresses $z = \langle x, y\rangle$:

$$(\exists z_1, z_2 \in z)\big(x \in z_1 \wedge (\forall w \in z_1)(w = x) \wedge \\ x \in z_2 \wedge y \in z_2 \wedge \wedge (\forall w \in z_2)(w = x \vee w = y) \wedge \\ (\forall w \in z)(w = z_1 \vee w = z_2)\big)$$

By collection, for any $x \in a$ there exists c s.t. $(\forall y \in b)(\exists z \in c)z = \langle x, y\rangle$. And thus by Δ_0 separation for any x there is $c = \{\langle x, y\rangle \mid y \in b\}$. The property $c = \{\langle x, y\rangle \mid y \in b\}$ in fact is Δ_0, thus there exists $d = \{\{\langle x, y\rangle \mid y \in b\} \mid x \in a\}$.
Applying union to d we get $a \times b$. □

Σ-formulas

The class of Σ-formulas consists of all formulas φ such that all occurrences of unbounded $\exists$-quantifiers are positive and all occurrences of unbounded $\forall$-quantifiers are negative.

Examples:
$(\forall y \in a)\forall x(\exists z \in x)(z \in y) \to (\forall y \in a)\exists x(y \in x)$ is a Σ-formula
$\forall x(x = x)$ and $\exists x(x \in a) \to a = b$ are **not** Σ-formulas.

Σ-formulas

Theorem

For any Σ-formula $\varphi(\vec{v})$ there is a Σ_1-formula $\varphi'(\vec{v})$ such that theory KPU *proves that*

$$\forall \vec{v}(\varphi(\vec{v}) \leftrightarrow \varphi'(\vec{v})).$$

Proof.

By pushing negations to the level of atomic formulas we transform a Σ-formula into a formula, built from $\wedge$, $\vee$, bounded quantifiers, $\exists$, atomic formulas and negated atomic formulas.
For φ of this kind we prove theorem by induction on the construction. The only non-trivial case is the case of φ starting with a bounded quantifier. For example consider φ of the form $(\forall x \in a)\psi$. It is equivalent to $(\forall x \in a)\exists \vec{u}\psi'$, where ψ' is Δ_0. Thus by Collection φ is equivalent to $\exists b(\forall x \in a)(\exists \vec{u} \in b)\psi'$. □

Σ-reflection

For a formula φ the formula φ^a is the result of replacement of unbounded quantifiers $\forall x$, $\exists b$,... with the bounded quantifiers $\forall x \in a$, $\exists b \in a$, ...

Theorem

Instances of the following Σ-reflection scheme are provable in KPU:

$$\varphi \to \exists a\, \varphi^a, \text{ where } \varphi \in \Sigma,\ a \notin FV(\varphi).$$

Proof.

Notice that for Σ-formulas φ and any a, b we have
$a \subseteq b \to \varphi^a \to \varphi^b$ and $\varphi^a \to \varphi$.
Using this we prove instances of Σ-reflection by induction on construction of Σ-formulas, where negations could be used only on the level of atomic formulas.
The only non-trivial case is the case of bounded universal quantifier that we handle using collection. □

Alternative axiomatization of KPU

Alternative axioms for KPU

1. $\forall z\ (z \in a \leftrightarrow z \in b) \rightarrow a = b$ (Extensionality)
2. $\exists b \forall x (x \in b \leftrightarrow x \in b \wedge \varphi(x))$, where φ is Δ_0 and $b \notin FV(\varphi)$ (Δ_0-Separation)
3. $\varphi \rightarrow \exists a\ \varphi^a$, where $\varphi \in \Sigma$, $a \notin FV(\varphi)$. (Σ-reflection)
4. $\exists x\ \varphi(x) \rightarrow \exists x\ (\varphi(x) \wedge \neg(\forall y \in x)\varphi(x))$ (Foundation)

Recursion on sets and ordinals

As usual a set a is called transitive if $(\forall b \in a) b \subseteq a$.
As usual ordinals are transitive sets consisting only of transitive sets.

$$S(\alpha) \stackrel{\text{def}}{=} \alpha \cup \{\alpha\}$$

Ordinal arithmetic

- $\alpha + \beta = \bigcup(\alpha \cup \{S(\alpha + \gamma) \mid \gamma < \beta\})$
- $\alpha\beta = \bigcup\{\alpha\gamma + \alpha \mid \gamma < \beta\}$
- $\alpha^{\beta} = \bigcup(\{S(0)\} \cup \{\alpha^{\gamma}\alpha \mid \gamma < \beta\})$

Theorem

The following is formalizable in KPU. *Suppose $f(\vec{x}, y, a)$ is a Σ_1-definable function. Then*

$$g(\vec{x}, y) = f(\vec{x}, y, \{\langle z, g(\vec{x}, z)\rangle \mid z \in y\})$$

is a Σ_1-definable function.

Transitive models of KP

Recall that a set A is called transitive if $(\forall b \in A) b \subseteq A$.

We treat transitive sets A as models $(A, \in)$. Models of this form are called transitive models.

$\mathbb{HF}$ is the least transitive model of KP.

Theory KPω is the extension of KP by the the axiom of infinity, i.e. the assertion that ordinal ω exists. The least transitive model of KPω is $L_{\omega_1^{CK}}$.

Transitive models of KPU

Fix a model $\mathfrak{M} = (M, R_1, \dots, R_n)$.
A transitive model over $\mathfrak{M}$ given by a transitive set A, $M \subseteq A$ is $(M, A; \in, R_1, \dots, R_n)$.

Transitive models of KP/KPU are called admissible sets.

$\mathbb{HF}(\mathfrak{M})$ is the least admissivle set over $\mathfrak{M}$.

Let KPU$^+$ be KPU $+ \exists a \forall p (p \in a)$. The last axiom states that there exists the set of all urelements. Transitive models of KPU$^+$ over $\mathfrak{M}$ are called admissible sets above $\mathfrak{M}$.

The least admissible set above $\mathfrak{M}$ is denoted as $\mathbb{HYP}(\mathfrak{M})$.

Constructible models of KPU

- $L_0(\mathfrak{M}) = \mathfrak{M}$
- $L_{\alpha+1}(\mathfrak{M}) = \mathfrak{M} \cup \{a \subseteq L_\alpha \mid$
  a is definable in the transitive model $L_{\alpha+1}(\mathfrak{M})\}$
- $L_\lambda(\mathfrak{M}) = \bigcup_{\beta<\lambda} L_\beta(\mathfrak{M})$, for limit ordinals λ.

For an admissible set A let $o(A) = \sup\{\alpha \mid \alpha \in \mathrm{On} \cap A\}$.

Theorem

For any admissible set A above $\mathfrak{M}$ the model $L_{o(A)}(\mathfrak{M})$ is an admissible set above $\mathfrak{M}$ and $L_{o(A)}(\mathfrak{M}) \subseteq A$.

Theorem

$\mathbb{HYP}(\mathfrak{M}) = L_\alpha(\mathfrak{M})$ *for the least α such that $L_\alpha(\mathfrak{M}) \models \mathrm{KPU}^+$.*

Ill-founded models of KPU

Suppose $\mathfrak{A}$ is a model of KPU^+ above $\mathfrak{M}$, where $\in^{\mathfrak{A}}$ is not necessarily the standard $\in$.
Let $\mathrm{WF}(\mathfrak{A})$ be the well-founded part of $\mathfrak{A}$ i.e. the submodel of $\mathrm{WF}(\mathfrak{A})$ that contains all elements x such that there are no infinite chain

$$x = x_0 \ni^{\mathfrak{A}} x_1 \ni^{\mathfrak{A}} x_2 \ni^{\mathfrak{A}} \dots$$

Theorem

For any model $\mathfrak{A} \models \mathrm{KPU}^+$ above $\mathfrak{M}$, the model $\mathrm{WF}(\mathfrak{A}) \models \mathrm{KPU}^+$ and is isomorphic to an admissible A above $\mathfrak{M}$.

Corollary

For any $\mathfrak{A} \models \mathrm{KPU}^+$ above $\mathfrak{M}$ we have $\mathbb{HYP}(\mathfrak{M}) \subseteq_{\mathrm{end}} \mathfrak{A}$.

Π^1_1 vs Σ_1

Recall that Π^1_1 in the signature $R_1, \ldots, R_n$ is the class of all second-order formulas of the form

$$\forall P_1^{(r_1)}, \ldots, P_m^{(r_m)} \varphi(P_1^{(r_1)}, \ldots, P_m^{(r_m)}),$$

where φ is a first-order formula with additional predicates $P_i^{(r_i)}(x_1, \ldots, x_{r_i})$.

Theorem

For countable models $\mathfrak{M} = (M, R_1, \ldots, R_n)$ a set $H \subseteq M^k$ is Π^1_1-definable in $\mathfrak{M}$ iff H is Σ_1 in $\mathbb{HYP}(\mathfrak{M})$. The "if" part holds even for uncountable $\mathfrak{M}$.

A set $H \subseteq M^k$ is called Δ^1_1-definable if both it and its complement are Π^1_1-definable

Corollary

For countable models $\mathfrak{M} = (M, R_1, \ldots, R_n)$ a set $H \subseteq M^k$ is Δ^1_1-definable iff $H \in \mathbb{HYP}(\mathfrak{M})$.

Recursively saturated models

A model $\mathfrak{M}$ is called recursively saturated if for any computable set of formulas Φ depending on variables $\vec{x}$ if for any finite subset $\Phi' \subseteq \Phi$ we have $\mathfrak{M} \models \exists \vec{x} \bigwedge_{\varphi \in \Phi'} \varphi(\vec{x})$, then there is $\vec{p} \in \mathfrak{M}$ such that $\mathfrak{M} \models \varphi(\vec{p})$, for any $\varphi(\vec{x})$ from Φ.

Theorem (Barwise, Schlipf)

For any $\mathfrak{M}$ the following are equivalent:

1. *$\mathfrak{M}$ is recursively saturated,*
2. *$o(\mathbb{HYP}(\mathfrak{M})) = \omega$.*

KPω and subsystems of PA_2

In KPω we naturally could interpret language of second-order arithmetic. Natural numbers are interpreted by finite ordinals and sets of natural by subsets of ω.
We have the following correspondences:

1. $\mathsf{KP}\omega \vdash \mathsf{ACA}_0$
2. $\mathsf{KP}\omega \vdash \Sigma^1_1\text{-}\mathsf{AC}_0$
3. $\mathsf{KPi} = \mathsf{KP}\omega + \forall x \exists y (x \in y \land y \models \mathsf{KP}\omega)$ has the same second-order consequences as $\Delta^1_2\text{-}\mathsf{CA}_0 + \mathsf{BI}$
4. $\mathsf{KP}\omega + \Sigma_1$-Separation has the same second-order consequences as $\Pi^1_2\text{-}\mathsf{CA}_0 + \mathsf{BI}$

Thank you!