On the (non)existence of proof sytems in universal proof theory

(in 3 chapters)

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Chapter 1

Proof Theory

1.1 Origins of Proof Theory

In 1900 a the *International Congress of Mathematicians* David Hilbert posed 23 problems:

2 The consistency of the axioms of arithmetic.

The proposal for 2 is via proof-theoretic methods:

To conquer this field [concerning the foundations of mathematics] we must turn the concept of a specifically mathematical proof itself into an object of investigation, just as the astronomer considers the movement of his position, the physicist studies the theory of his apparatus, and the philosopher criticizes reason itself.

Hilbert's Program:

Proof systems as rigorous axiomatization/representation of (an area in) mathematics.

Proving consistency (of the area) by showing that the proof system cannot derive falsities in a theory which consistency is beyond doubt.

Thm (Kurt Gödel - announced 1930, published 1931)
No sufficiently strong first-order theory can prove its own consistency.

Considered to be the end of Hilbert's Program:

If a theory cannot prove its own consistency, how can it prove the consistency of a stronger theory?

Not the end of Proof Theory:

Relative consistency; rigour; faithfully formalize reasoning . . .

Chapter 1

Proof Theory

1.2 Natural proofs

Early days of Proof Theory: proof systems were Hilbert systems.

Ex Classical propositional logic CPC for language $\{\neg, \to\}$ has proof system H_1:

Axioms:

$$\varphi \to (\psi \to \varphi)$$

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

Rule:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \text{ Modus Ponens (MP)}$$

Ex A proof in H_1:

$$\frac{\varphi \to (\psi \to \varphi) \qquad (\varphi \to (\psi \to \varphi)) \to ((\varphi \to \psi) \to (\varphi \to \varphi))}{(\varphi \to \psi) \to (\varphi \to \varphi)} \ (\mathrm{MP})$$

Not a natural proof system: Try proof $(\varphi \to \varphi)$ in H_1 ...

Another axiomatization H_2: Rule MP plus axiom

$$((((\varphi_1 \to \psi_1) \to (\neg\varphi_2 \to \neg\psi_2)) \to \varphi_2) \to \varphi_3) \to ((\varphi_3 \to \varphi_1) \to (\psi_2 \to \varphi_1))$$

Gerhard Gentzen in the 1930s developed *natural deduction* to "set up a formalism that reflects as accurately as possible the actual logical reasoning involved in mathematical proofs".

A proof in natural deduction (ND) starts from assumptions and derives further statements by "natural rules".

Ex

$$\frac{\varphi_1 \qquad \varphi_2}{\varphi_1 \wedge \varphi_2}\, I\wedge \qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_i}\, E\wedge \text{ (i=1,2)} \qquad \frac{\begin{matrix}\vdots \\ \bot\end{matrix}}{\varphi}\text{ Ex Falso}$$

The meaning of the logical connectives is expressed in the rules for the connective. Gerhard Gentzen (1934/35) *Untersuchungen über das logische Schließen*:

The introductions represent, as it were, the "definitions" of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.

Def A proof is in *normal form* if it is does not contain detours.

Ex A possible detour:

$$\dfrac{\dfrac{\varphi_1 \qquad \varphi_2}{\varphi_1 \wedge \varphi_2}\, I\wedge}{\varphi_1}$$

Normal proofs are desirable for many reasons.

Thm (Gentzen)
Intuitionistic logic has normalization (every provable formula has a proof in normal form).

Thm (Prawitz 1965, van Plato & Siders 2012) Classical logic has normalization.

To prove normalization for classical logic, Gentzen developed the sequent calculus.

The beauty and usefulness of sequent calculi has made them primary proof systems, especially in universal and structural proof theory.

For Γ a finite multiset (of assumptions):

$$\begin{array}{c}[\Gamma] \\ \vdots \\ \hline \varphi\end{array}$$

in natural deduction becomes

$$\Gamma \vdash \varphi$$

is generalized to

$$\Gamma \vdash \Delta$$

becomes in sequent calculus

$$\Gamma \Rightarrow \Delta$$

In second step, conclusion φ is replaced by a finite multiset of formulas Δ.

Chapter 1

Proof Theory

1.3 Sequent calculi

Def Language $\mathcal{L}$ of *classical propositional logic* CPC:

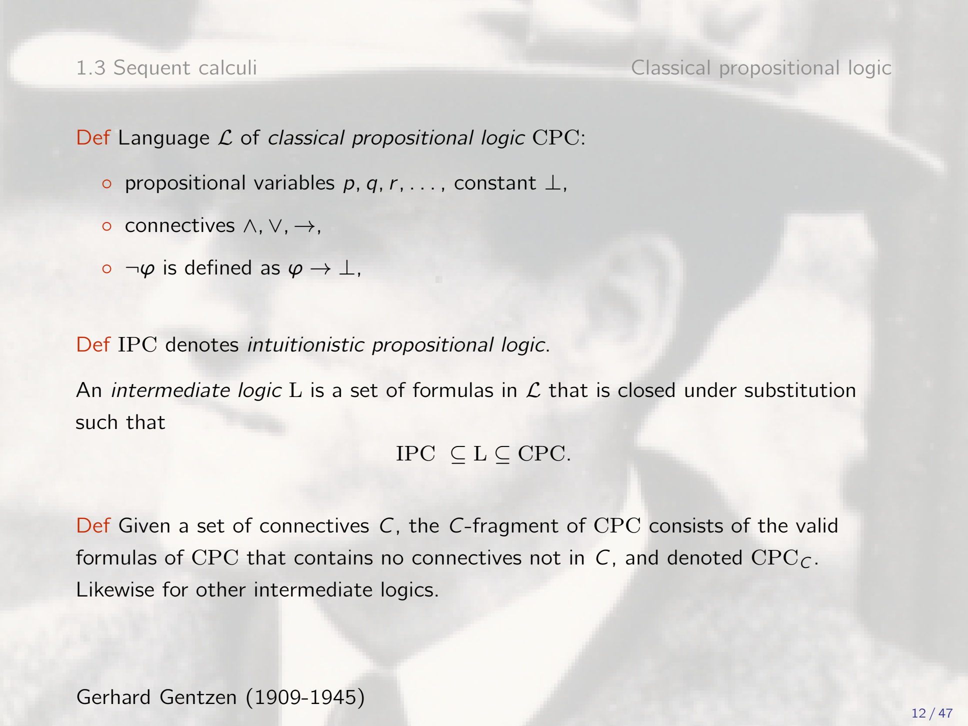
- propositional variables $p, q, r, \ldots$, constant $\bot$,
- connectives $\wedge, \vee, \to$,
- $\neg\varphi$ is defined as $\varphi \to \bot$,

Def IPC denotes *intuitionistic propositional logic*.

An *intermediate logic* L is a set of formulas in $\mathcal{L}$ that is closed under substitution such that

$$\mathrm{IPC} \subseteq \mathrm{L} \subseteq \mathrm{CPC}.$$

Def Given a set of connectives C, the C-fragment of CPC consists of the valid formulas of CPC that contains no connectives not in C, and denoted CPC_C. Likewise for other intermediate logics.



Gerhard Gentzen (1909-1945)

Def A *sequent* is an expression $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas.

{ } often omitted:
$\varphi_1, \ldots, \varphi_m \Rightarrow \psi_1, \ldots, \psi_n$ instead of $\{\varphi_1, \ldots, \varphi_m\} \Rightarrow \{\psi_1, \ldots, \psi_n\}$.

Interpretation: $I(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma \to \bigvee \Delta)$.

Ex

$$I(\varphi, \psi \Rightarrow \varphi, \chi, \chi) = (\varphi \wedge \psi \to \varphi \vee \chi \vee \chi)$$

$$I(\neg\varphi \Rightarrow \varphi \to \psi) = (\neg\varphi \to (\varphi \to \psi))$$

Def A *sequent calculus (SC)* G is a set of rules, where a *rule* R is an expression of the form (the S_i are sequents):

$$\frac{S_1 \quad \ldots \quad S_n}{S_0}\, R$$

It is an *axiom* if the set of premises is empty: $\quad S_0$.

Ex Sequent calculus $\mathrm{G}_\vee$ for $\mathrm{CPC}_\vee$:

Axioms: Ax_{at} $p \Rightarrow p$ $Ax_\bot$ $\bot \Rightarrow$

Rules for the logical operators:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\, R\vee$$

Structural rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\, LW \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\, RW \quad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\, LC \quad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\, RC$$

Def A *proof* of S in a sequent calculus G: a finite tree of sequents with root S, the sequents at the leafs are axioms of G, and for every node S' that is not a leaf, there is an instance $R = (S_1, \ldots, S_n/S')$ of a rule in G such that the premises $S_1, \ldots, S_n$ are exactly the immediate predecessors of S' in the tree.

Ex
$$\cfrac{\cfrac{p \Rightarrow p}{p \Rightarrow p, q}\, RW}{p \Rightarrow p \vee q}\, R\vee \qquad \cfrac{\cfrac{\cfrac{\cfrac{p \Rightarrow p}{p \Rightarrow p, q, p, q}\, RW^*}{p \Rightarrow p \vee q, p, q}\, R\vee}{p \Rightarrow p \vee q, p \vee q}\, R\vee}{p \Rightarrow p \vee q}\, RC$$

Def Sequent calculus G1cp for CPC:

Axioms: Ax_{at} $\quad p \Rightarrow p$ $\qquad Ax_{\bot}$ $\quad \bot \Rightarrow$

Structural rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ RW \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ RC$$

Rules for the logical operators:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ R\vee$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ R\wedge$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ L\to \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ R\to$$

Ex

$$\cfrac{\cfrac{\cfrac{p \Rightarrow p}{p \Rightarrow p, \bot}\ RW}{\Rightarrow p, \neg p}\ R\to}{\Rightarrow p \vee \neg p}\ R\vee$$

Def Sequent calculus G1cp:

Axioms: Ax_{at} $p \Rightarrow p$ $Ax_{\bot}$ $\bot \Rightarrow$

Structural rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ RW \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ RC$$

Rules for the logical operators:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ R\vee$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ R\wedge$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ L\to \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ R\to$$

Thm G1cp is sound and complete for CPC: For any formula φ,

$(\Rightarrow \varphi)$ is derivable in G1cp if and only if φ is valid in CPC.

Note Proof search in G1cp is not terminating due to the structural rules.

Def $\mathrm{G1ip}$ is $\mathrm{G1cp}$ in which every sequent has at most one formula on the right and $L\to$ and $R\vee$ are replaced by

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ {\scriptstyle L\to} \qquad \frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2}\ {\scriptstyle R\vee} \quad (i=1,2)$$

Beautiful fact:

Thm $\mathrm{G1ip}$ is sound and complete for intuitionistic propositional logic IPC.

Ex The following derivation in $\mathrm{G1cp}$ cannot be carried out in $\mathrm{G1ip}$:

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow p, \bot}\ {\scriptstyle RW}}{\Rightarrow p, \neg p}\ {\scriptstyle R\to}}{\Rightarrow p \vee \neg p}\ {\scriptstyle R\vee}$$

Indeed, $I(\Rightarrow p \vee \neg p)$, equivalent to $p \vee \neg p$, does not hold in IPC.

Chapter 2

Calculi for a multitude of logics

2.1 Logics and their sequent calculi

Def Language $\mathcal{L}_\Box$ of *classical modal logic* is $\mathcal{L}$ plus a modal operator $\Box$.

Modal logics occur in

philosophy	$\Box\varphi$: agent knows φ, or φ is necessary
computer science	$\Box\varphi$: after any run of the program φ holds
mathematics	$\Box\varphi$: φ is a provable
$\vdots$	

More expressive than propositional logics, yet good computational properties.

Modal logics are usually axiomatized over CPC by some modal principles.
Ex Some well-known principles:

(K) $\quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(4) $\quad \Box\varphi \to \Box\Box\varphi$

(T) $\quad \Box\varphi \to \varphi$

(D) $\quad \neg\Box\bot$

$$\frac{\varphi}{\Box\varphi} \text{ Necessitation Rule (NR)}$$

Def The modal logic K is axiomatized over CPC by rule (NR) and principle (K).

Modal logics		Their G1 sequent calculus	
K	$\mathrm{CPC} + \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	G1K	$\mathrm{G1cp} + \dfrac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} R_K$
K4	$\mathrm{K} + \Box\varphi \to \Box\Box\varphi$	G1K4	$\mathrm{G1cp} + \dfrac{\Box\Gamma, \Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} R_4$
KD	$\mathrm{K} + \neg\Box\bot$	G1KD	$\mathrm{G1K} + \dfrac{\Gamma, \varphi \Rightarrow}{\Box\Gamma, \Box\varphi \Rightarrow} R_D$

Lemma $\mathrm{G1K} \vdash \Box(\varphi \to \psi) \Rightarrow \Box\varphi \to \Box\psi$.

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi, \psi} RW \quad \dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} LW}{\varphi \to \psi, \varphi \Rightarrow \psi} L\to}{\Box(\varphi \to \psi), \Box\varphi \Rightarrow \Box\psi} R_K}{\Box(\varphi \to \psi) \Rightarrow \Box\varphi \to \Box\psi} R\to$$

Lemma $\mathrm{G1KD} \vdash\ \Rightarrow \neg\Box\bot$.

Proof (Recall $\neg\Box\bot = \Box\bot \to \bot$)

$$\dfrac{\dfrac{\dfrac{\bot \Rightarrow}{\Box\bot \Rightarrow} R_D}{\Box\bot \Rightarrow \bot} RW}{\Rightarrow \neg\Box\bot} R\to$$

Chapter 2

Calculi for a multitude of logics

2.2 Results about sequent calculi

Early proof theory focussed on systems for strong theories where the details of the systems are less important.

Contemporary proof theory adds to this the investigation of proof systems in terms of their qualities, both practical, theoretical, and conceptual.

Different proof-theoretic aims correspond to different calculi.

We will see: G1 systems are faithful and valuable for proof generation. Other calculi, the $\mathrm{G3}$ systems, are usful for proof search, among other things.

The key difference between the calculi lies in the structural rules.

Reasons in favor of structural rules:

For the faithful formalization of “real” proofs another structural rule is needed:

A key structural rule: (φ is the *cut-formula*):

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Cut$$

In math proofs, the statement that ψ_1 implies ψ_2 is often proved by finding a lemma (φ) such that ψ_1 implies φ and φ implies ψ_2. Cut captures that inference:

$$\frac{\psi_1 \Rightarrow \varphi \qquad \varphi \Rightarrow \psi_2}{\psi_1 \Rightarrow \psi_2}\ Cut$$

Without Cut the proof of $\psi_1 \Rightarrow \psi_2$ cannot be obtained directly from the proofs of the premises.

$\mathrm{G1cp} + \mathrm{Cut}$ admits using lemma’s in proofs.

Important: $\mathrm{G1cp}$ does not prove more than $\mathrm{G1cp} + \mathrm{Cut}$, but it allows more proofs.

The G3 systems are sequent calculi without structural rules that are equivalent to their G1 counterpart.

Def G3cp is G1cp minus structural rules, in which the axioms are replaced by

$$Ax_{at} \quad \Gamma, p \Rightarrow p, \Delta \qquad\qquad Ax_{\bot} \quad \Gamma, \bot \Rightarrow \Delta$$

Thm (Cut-elimination theorem)
G3cp is equivalent to G1cp + Cut.

Cor G3cp is sound and complete w.r.t. CPC.

Note Proof-search in G3cp is terminating.

Reasons against structural rules:

For some applications structural rules are a burden:

- Proof search in $\mathrm{G1cp}$ and $\mathrm{G1ip}$ is not terminating due to the structural rules.
- Derivations don't have the subformula property.
- A proof of the decidability of the logics via their calculi is not straightforward.
- Likewise for other properties of the logics, e.g. the disjunction property of IPC.

Structural rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\, LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\, RW \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\, LC \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\, RC$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Cut$$

Look for calculi with good computational properties: the G3 systems.

Given a calculus $\mathrm{G} + \mathrm{Cut}$ that has cut-elimination (is equivalent to G) what does it cost to remove cuts from a proof in $\mathrm{G} + \mathrm{Cut}$?

Thm (Gentzen) Given a derivation $\mathcal{D}$ of sequent S in $\mathrm{G3c} + \mathrm{Cut}$ in which the largest cutformula has size n, there exists a derivation $\mathcal{D}'$ of S in $\mathrm{G3c}$ such that $|\mathcal{D}'| \leq 2_n^{|\mathcal{D}|}$.

($2_n^{|\mathcal{D}|}$ is a tower of 2s of height n, where the highest 2 has power $|\mathcal{D}|$, the length of the longest branch in $\mathcal{D}$.)

Thm (Orevkov, Statman) The above bound is sharp: There are sequents S_m that have linear proofs in $\mathrm{G3c} + \mathrm{Cut}$ but which cut-free proof in $\mathrm{G3c}$ requires size at least 2_m.

Conclusion: Removing cuts is costly.

Similar results hold for other logics, e.g. intuitionistic predicate logic $\mathrm{G3i}$.

General pattern for many logics L:

- L has a G1 system $\mathrm{G1L}$, and a G3 system $\mathrm{G3L}$ without structural rules;
- $\mathrm{G1L} + \mathrm{Cut}$ and $\mathrm{G3L}$ are equivalent;
- $\mathrm{G1L}+ \mathrm{Cut}$ is a relatively faithful/natural proof system for reasoning in L;
- $\mathrm{G3L}$ has good computational properties.
- $\mathrm{G3L}$ is analytic, it has the *subformula property*: Every formula in a derivation is a subformula of a formula in the endsequent.
- All derivations in $\mathrm{G3L}$ are *pure*.

Ex

- $\mathrm{L} = \mathrm{CPC}$ and $\mathrm{G1cp} + \mathrm{Cut}$ and $\mathrm{G3cp}$.
- $\mathrm{L} = \mathrm{IPC}$ and $\mathrm{G1ip} + \mathrm{Cut}$ and $\mathrm{G3ip}$.
- $\mathrm{L} = \mathrm{K}$ and $\mathrm{G1K} + \mathrm{Cut}$ and $\mathrm{G3K}$.

$$\vdots$$

The hard part: finding good calculi $\mathrm{G1L} + \mathrm{Cut}$ and $\mathrm{G3L}$ and proving that they are equivalent (the cut-elimination theorem).

Chapter 3

The existence of sequent calculi

3.1 The usefulness of sequent calculi

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Sequent calculi can be very useful. E.g. to prove for a logic

- decidability;
- interpolation;
- translation into other logics;
- Herbrand's theorem;

$\vdots$

Question: Has a given logic L a sequent calculus?

Answer: Yes. Add to $\mathrm{G1cp}$ the rule Cut and axioms ($\Rightarrow \varphi$) for all theorems φ of L. The result is a calculus for L.

But not with good computational properties. It is not a $\mathrm{G3}$ calculus without structural rules.

Question: Has a given logic L a *good* sequent calculus?

What makes a sequent calculus G *good* depends on its applications.

Ex Notions of *good* in $\mathrm{G3}$ style:

- structural rules are admissible in G
- G is cut-free
- G has the subformula property
- G is finite
- proof search is (semi-)terminating in some order on sequents

Def In this talk: G is *good* if G is *terminating* and an extension of $\mathrm{G3cp}$ ($\mathrm{G3ip}$) by *semi-analytic* (single-conclusion) rules and *focused* axioms (to be defined).

Def G is *terminating* if

finite G is finite;

instance finite for every sequent S there are at most finitely many instances of rules in G with conclusion S;

well-ordered there is a well-order $\prec$ on sequents such that

- every proper subsequent comes before the sequent in $\prec$;
- in all rules in G, the premises come before the conclusion in $\prec$.

Ex $\mathrm{G3cp}$ is a terminating calculus for CPC, and $\mathrm{G3ip}$ for IPC.

Ex The $\mathrm{G3}$ systems for modal logics K, KT, KD from Chapter 2 are terminating calculi. And so are the $\mathrm{G4i}$ systems for intuitionistic modal logics $\mathrm{iK}_\Box$, $\mathrm{iKD}_\Box$.

Note Good calculi are very useful in investigating logics.

The Question: Which logics do (not) have good sequent calculi?

Similar questions in the literature:

Mostly on cut-free seqent calculi and structural rules.

E.g. the work by Belardinelli & Jipsen & Ono, later extended by Ciabattoni & Galatos & Terui, on the existence of cut-free sequent calculi.

E.g. the work by Negri on labelled sequent calculi.

$\vdots$

The Question: Which logics do (not) have good sequent calculi?

There are

- numerous positive results of the form:

  These logics have a good sequent calculus

- few(er) negative results of the form:

  These logics do not have a good sequent calculus.

The Question: Which logics do (not) have good sequent calculi?

There are many many results of the form:
this logic (the logics in this class) has a good sequent calculus.

Some well-known members of these classes have good sequent calculi:

- modal logics,
- non-normal modal logics,
- intermediate logics,
- intuitionistic modal logics,
- conditional logics,
- substructural logics.

The Question:

Which logics do not have a good sequent calculus?

Method:

Consider a class of logics $\mathcal{C}$ (intermediate, modal, etc). Suppose there is a subclass $\mathcal{C}' \subseteq \mathcal{C}$ and a property $\mathcal{P}$ such that one can show:

(I) Any logic in $\mathcal{C}$ with a good calculus has property $\mathcal{P}$.

(II) No logic in $\mathcal{C}'$ has property $\mathcal{P}$.

Then by easy contraposition:

(III) No logic in $\mathcal{C}'$ has a good calculus.

The strength of the method depends on $\mathcal{P}$ being rare and $\mathcal{C}'$ being large.

Next slides: Method will be demonstrated for $\mathcal{C}$ being the class of intermediate logics and $\mathcal{P}$ being uniform interpolation.

Def A calculus G for intermediate logics is *good* if G is terminating and an extension of $\mathrm{G4ip}$ by semi-analytic single-conclusion rules and focused axioms.

Def A *focused axiom* is an axiom of the form

$$\Gamma, \varphi_1, \dots, \varphi_n \Rightarrow \Delta \qquad \Gamma \Rightarrow \varphi.$$

Def A *semi-analytic rule* is of one of the following the forms:

$$\frac{[\Gamma_i, \bar{\psi}_i \Rightarrow \chi_i]_i}{\Gamma_1, \dots, \Gamma_n \Rightarrow \varphi} \qquad \frac{[\Gamma_i, \bar{\psi}_i \Rightarrow \chi_i]_i \qquad [\Pi_i, \bar{\theta}_i \Rightarrow \Delta_i]_i}{\Gamma_1, \dots, \Gamma_m, \Pi_1, \dots, \Pi_n, \varphi \Rightarrow \Delta_1, \dots, \Delta_n}$$

(This is a simplified version of the definition in the literature.)

Ex Rule $R_{\rightarrow}$ is semi-analytic, Cut is not.

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi}\, R_{\rightarrow} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Cut$$

Fact Many rules in standard sequent calculi are semi-analytic.

Method to prove negative results for intermediate logics:

Suppose there is a large subclass $\mathcal{C}'$ of intermediate logics and a property $\mathcal{P}$ such that one can show:

(I) Any intermediate logic with a good calculus has property $\mathcal{P}$.

(II) No logic in $\mathcal{C}'$ has property $\mathcal{P}$.

Then:

(III) No intermediate logic in $\mathcal{C}'$ has a good calculus.

Next slides: Method will be demonstrated for $\mathcal{P}$ being uniform interpolation.

Def A logic L has (Craig) *interpolation (CIP)* if whenever $\vdash \varphi \to \psi$ there is a χ in the common language $\mathcal{L}(\varphi) \cap \mathcal{L}(\psi)$ such that $\vdash \varphi \to \chi$ and $\vdash \chi \to \psi$.

Def A logic L has *uniform interpolation (UIP)* if the interpolant depends only on the premise or the conclusion: For all atoms p and formulas φ there are formulas, denoted $\exists p\varphi$ and $\forall p\varphi$, in the language of L that do not contain p or any variable not in φ, such that for all ψ not containing p:

$$\vdash \psi \to \varphi \text{ iff } \vdash \psi \to \forall p\varphi \qquad \vdash \varphi \to \psi \text{ iff } \vdash \exists p\varphi \to \psi.$$

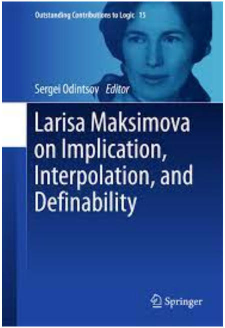
Ex $\vdash \varphi \to \exists p\varphi \qquad \vdash \forall p\varphi \to \varphi \qquad \exists q((p \to q) \wedge \neg q) = \neg p.$

Def A sequent calculus G has *uniform sequent interpolation (USIP)* if for any sequent $S = (\Gamma \Rightarrow \Delta)$ there exist p-free formulas $\exists pS$ and $\forall pS$ s.t. for all p-free Π, Σ:

$$\vdash \Gamma \Rightarrow \exists pS, \Delta \qquad \vdash \Gamma, \forall pS \Rightarrow \Delta$$

$$\vdash \Gamma, \Pi \Rightarrow \Delta, \Sigma \text{ implies } \vdash \Pi, \exists pS \Rightarrow \forall pS, \Sigma.$$

Lemma If G is a calculus for logic L with USIP, then L has UIP.



Thm (Pitts '92) IPC has uniform interpolation.

Thm Modal logics with uniform interpolation:

- GL (Shavrukov '94)
- K (Ghilardi & Zawadowski '95, Visser '96)
- KT (Bilkova '06).

Thm Modal logics without uniform interpolation:

- S4 (Ghilardi & Zawadowski '95)
- K4 (Bilkova '06).

Thm (Maksimova '77, Ghilardi & Zawadowski '02)

There are exactly seven intermediate logics with (uniform) interpolation: IPC, Sm, GSc, LC, KC, Bd_2, CPC.

Thm (Maksimova '80s & '90s)
Among the normal extensions of S4 there are at least 31 and at most 49 logics with interpolation. Exactly 7 normal extensions of Grz have interpolation.

Thm If an intermediate logic has a good calculus G, then it has uniform interpolation.

Proof pitch Define the uniform interpolant (uip) based on the rules of G (using that G is good). Then show for any rule $S_1 \dots S_n/S$ that if all S_i have a uip, then so does S (using that G is semi-analytic). ⊣

Thm (I. 2017, strengthened by Jalali & Tabatabai 2018)
Any intermediate logic with a good calculus has uniform interpolation.

(The calculi are extensions of Dyckhoff's $\mathrm{G4i}$.)

Thm (Maksimova '77, Ghilardi & Zawadowski '02) There are exactly seven intermediate logics with (uniform) interpolation.

Cor No intermediate logic except those 7 can have a good calculus.

Positive corollaries:
Cor (well-known) Classical propositional logic CPC has uniform interpolation, as all rules in $\mathrm{G3c}$ are semi-analytic.

Thm (for substructural logics, Tabatabai & Jalali 2018)
Any logic with a terminating sequent calculus that extends the standard calculus for FL_{e} and consists of semi-analytic rules has uniform interpolation.

Thm (I. 2016, strengthened by Jalali & Tabatabai 2018)
Any normal modal logic with a good calculus has uniform interpolation.

Thm (Maksimova '80s & '90s) Among the normal extensions of $\mathrm{S4}$ there are at least 31 and at most 49 logics with interpolation. Exactly 7 normal extensions of Grz have interpolation.

Cor No normal extension of $\mathrm{S4}$ or Grz, except those 49 and 7 respectively, can have a good calculus.

Cor $\mathrm{K4}$ and $\mathrm{S4}$ do not have good calculi.

Although not the aim, the method provides positive results too, namely proof-theoretic proofs of UIP, e.g. for K and KD:

Cor The modal logics K (Ghilardi) and KD (Pattinson) have uniform interpolation.

Cor(I. 2019) iK and iKD (without diamond) have uniform interpolation.

(I., Jalali, Tabatabai 2020 – 2022:)

Proving UIP for intuitionistic modal, conditional and non-normal modal logics based on sequent calculi that are variants of good calculi.

(Jalali, Tabatabai 2022)

A similar method for intermediate logics where UIP is replaced by the Disjunction Property and other admissible rules.

(I.)

Tightening the connection between terminating sequent calculi and UIP.

Question for this talk: Which logics do have a good proof system?

This talk: Providing an answer for some logics (intermediate & (intuitionistic) modal), some proof systems (sequent calculi) and some notions of *good* (terminating, semi-analytic rules).

General insight

Having a good sequent calculus implies uniform interpolation.

Large classes of modal and intermediate logics do not have good proof systems, since they do not have uniform interpolation.

By-product

Widely applicable constructive method to prove UIP (nonnormal modal, conditional).

Related questions:

For what other notions of "good" can we develop a similar method?

How to adapt the method to predicate logics?

Finis