Reflection algebras and progressions

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Standard axiomatic theories

Peano arithmetic PA: formalizes 'finitary mathematics'; based on axioms for natural numbers with $+$ and $\cdot$.

Second order arithmetic PA^2: formalizes analysis; extends PA by variables for sets of numbers and assumes the schemata of full comprehension and induction.

Zermelo–Fraenkel set theory ZFC: formalizes *all* conventional mathematics; based on axioms for sets and membership relation.

Formal axiomatic theories are materialized in various *automatic* and *interactive theorem provers* such as Coq, Isabelle/HOL or Mizar.

Comparing axiomatic theories

Theories differ in

- the expressivity of their languages (*richness*);
- the amount of axioms (*strength*),
- speed of proofs,
- deductive mechanism, etc.

We need to develop a systematic way to compare and measure strength of theories.

The Worm principle

Hamano, Okada (1997), Beklemishev (2002)

Consider words in the alphabet $\mathbb{N}$.
A word α *is higher than* n if each letter of α exceeds n.

Given α, generate the following sequence $(\alpha_n)_{n\in\omega}$ of words.

Set $\alpha_0 := \alpha$ and define α_{k+1} by the following two rules:

- If $\alpha_k = 0\beta$ then $\alpha_{k+1} := \beta$.
- If $\alpha_k = (n+1)\beta$, find the longest (possibly empty) prefix β_0 of β such that β_0 is higher than n. Assume $\beta = \beta_0\gamma$. Then let $\alpha_{k+1} := (n\beta_0)^{k+2}\gamma$.

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Example

$\alpha_0 = 13022$

$\alpha_1 = 0303022$

$\alpha_2 = 303022$

$\alpha_3 = 222203022$

$\alpha_4 = 1222122212221222122203022$

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Independence results

The *Worm principle* (WP) states that, for each α, the sequence α_k terminates in an empty word.

Theorem.

- *WP* is true but unprovable in Peano arithmetic;
- *WP* is equivalent in EA to the Σ_1-reflection $R_1(\mathrm{PA})$ for PA;
- The function $F(\alpha) := \mu n.(\alpha_n \text{ is empty})$ exceeds any computable function provably total in PA.

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Background formal arithmetic

Elementary arithmetic EA is formulated in the language $(0, 1, +, \cdot, 2^x, \leq, =)$ and has some minimal set of basic axioms defining these symbols plus the induction schema for bounded formulas.[1]

A formula is *bounded* if all its quantifier occurrences are of the form $\forall x \leq t$ or $\exists x \leq t$ where t is a term (not containing x).

Peano arithmetic PA is EA with full induction:

$$\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x),$$

where φ is any formula (possibly with parameters).

[1] EA is also known as $I\Delta_0 + \exp$ and EFA.

Quantifier complexity

Σ_n-formulas: $\exists x_1 \forall x_2 \ldots Q x_n \varphi(x_1, \ldots, x_n)$, with $\varphi(\vec{x})$ bounded.
Π_n-formulas: $\forall x_1 \exists x_2 \ldots Q x_n \varphi(x_1, \ldots, x_n)$

Fact. A set is Σ_1-definable in $\mathbb{N}$ iff it is recursively (computably) enumerable.

$I\Sigma_n = \mathrm{EA} +$ induction for Σ_n-formulas

$$\mathrm{EA} \subset I\Sigma_1 \subset I\Sigma_2 \cdots \subset \mathrm{PA}$$

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Gödel's 2nd Incompleteness Theorem

Definition

A theory T is Gödelian if

- Natural numbers and operations $+$ and $\cdot$ are definable in T;
- T proves basic properties of these operations (contains EA);
- There is an algorithm (and a Σ_1-formula) recognizing the axioms of T.

$\Box_T(x) =$ 'x is the Gödel number of a T-provable formula'
$\mathrm{Con}(T) =$ 'T is consistent'

K. Gödel (1931): If a Gödelian theory T is consistent, then $\mathrm{Con}(T)$ is true but unprovable in T.

A natural response to Gödel: add $\mathrm{Con}(T)$ to T as a new axiom.
Is $T + \mathrm{Con}(T)$ complete? No, because it is Gödelian.

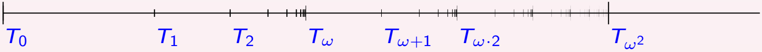
A. Turing (1939) suggested to continue the process:

$$T_0 = T$$
$$T_1 = T + \mathrm{Con}(T)$$
$$T_2 = T + \mathrm{Con}(T) + \mathrm{Con}(T + \mathrm{Con}(T))$$
$$\ldots$$
$$T_{n+1} = T_n + \mathrm{Con}(T_n)$$
$$\ldots$$

Is $\cup_{n\geq 0} T_n$ complete?

No: $T_\omega := \cup_{n\geq 0} T_n$ is Gödelian. Hence, T_ω does not prove $\mathrm{Con}(T_\omega)$ and the process continues:

$$T_{\omega+1} = T_\omega + \mathrm{Con}(T_\omega)$$
$$T_{\omega+2} = T_{\omega+1} + \mathrm{Con}(T_{\omega+1})$$
$$\ldots$$



Turing's classification program

Turing hoped to obtain a classification of all true arithmetical statements according to the stages of this (and similar) processes – but encountered difficulties.

A.M. Turing 1939 *System of logics based on ordinals*:

> *We might also expect to obtain an interesting classification of number-theoretic theorems according to "depth". A theorem which required an ordinal α to prove it would be deeper than one which could be proved by the use of an ordinal β less than α. However, this presupposes more than is justified.*

Turing progressions

The difficulties are:

- Logical complexity restriction;
- The problem of canonicity of ordinal notations.

Ordinal notations

Orderings can be represented in T, for example, by assigning rational numbers to points. The resulting set of numbers must be recognizable by an algorithm. (Otherwise, the axioms of T_α would not be recognizable.)

A problem: theories T_α depend on a particular way the ordering is computed rather than on the isomorphism type (the ordinal) of α.

Turing, Feferman, Kreisel: the whole classification idea breaks down because of this problem.

Turing's theorem

Theorem

For each true Π_1-sentence π there is a ordinal notation α such that $|\alpha| = \omega + 1$ and T_α proves π.

A. Turing:

> *This completeness theorem as usual is of no value. Although it shows, for instance, that it is possible to prove Fermat's last theorem with Λ_P (if it is true) yet the truth of the theorem would really be assumed by taking a certain formula as an ordinal formula*[2].

A partial way out: Careful selection of 'canonical' or 'natural' ordinal notations. This is possible for very large constructive ordinals, but we lack a general understanding of what is a natural ordinal notation system.

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Logical complexity restriction

Fact: There are true statements that cannot be proved at any stage of a Turing progression. Let

$$T' = T + \{\mathrm{Con}(S) : S \text{ any consistent Gödelian theory}\}.$$

T' obviously contains any T_α.

Is T' Gödelian? No: there is no algorithm to recognize the consistency of an arbitrary given system S.

Nonetheless, Gödel theorem holds for T': $\mathrm{Con}(T')$ is expressible but not provable in T'. Since $T_\alpha \subseteq T'$, T_α does not prove $\mathrm{Con}(T')$.

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Reflection principles

Let T be a Gödelian theory.

- *Reflection principles* $R_n(T)$ for T are arithmetical sentences expressing *"every* Σ_n*-sentence provable in* T *is true"*.

$R_n(T)$ can be seen as a relativization of the consistency assertion:

$R_0(T) \leftrightarrow \mathrm{Con}(T)$
$R_n(T) \leftrightarrow \mathrm{Con}(T + \text{all true } \Pi_n\text{-sentences})$

$R_n(T)$ is expressible as a Π_{n+1}-sentence.

Def. $\mathfrak{G}_{\mathsf{EA}}$ is the set of all Gödelian extensions of EA mod $=_{\mathsf{EA}}$.

$S \leq_{\mathsf{EA}} T \iff \mathsf{EA} \vdash \forall x\, (\Box_T(x) \to \Box_S(x))$;

$S =_{\mathsf{EA}} T \iff (S \leq_{\mathsf{EA}} T$ and $T \leq_{\mathsf{EA}} S)$.

Then $(\mathfrak{G}_{\mathsf{EA}}, \wedge_{\mathsf{EA}})$ is a lower semilattice with
$S \wedge_{\mathsf{EA}} T := S \cup T$
(defined by the disjunction of the formulas defining sets of axioms of S and T)

Monotone operators

Each of R_n correctly defines a monotone operator $R : \mathfrak{G}_S \to \mathfrak{G}_S$ on the semilattice of Gödelian extensions of S.

An operator R is:

- *monotone* if $x \leq y$ implies $R(x) \leq R(y)$;
- *semi-idempotent* if $R(R(x)) \leq R(x)$;
- *closure* if R is m., s.i. and $x \leq R(x)$.

All R_n are monotone and semi-idempotent, but not closure.

Iteration theorem

Def. $R : \mathfrak{G}_T \to \mathfrak{G}_T$ is *computable* if it can be defined by a computable map on the Gödel numbers of formulas defining the extensions of T.

Suppose $(\Omega, \prec)$ is an elementary recursive well-ordering and R is a computable m.s.i. operator on $\mathfrak{G}_T$.

Theorem

There exist theories $R^\alpha(S)$ (where $\alpha \in \Omega$):
$R^0(S) =_T S$ and, if $\alpha \succ 0$,

$$R^\alpha(S) =_T \bigcup\{R(R^\beta(S)) : \beta \prec \alpha\}.$$

Each R^α is computable and m.s.i.. Under some natural additional conditions the family R^α is unique modulo provable equivalence.

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Proof-theoretic ordinals and spectra

Let S be a Gödelian extension of EA and $(\Omega, <)$ a (natural) elementary recursive well-ordering.

- Π^0_{n+1}*-ordinal of* S, denoted $ord_n(S)$, is the sup of all $\alpha \in \Omega$ such that $S \vdash R_n^\alpha(\mathrm{EA})$;
- *Conservativity spectrum of* S is the sequence $(\alpha_0, \alpha_1, \alpha_2, \dots)$ such that $\alpha_i = ord_i(S)$.

Examples of spectra:

$I\Sigma_1: \ (\omega^\omega, \omega, 1, 0, 0, \dots)$

$\mathrm{PA}: \ (\varepsilon_0, \varepsilon_0, \varepsilon_0, \dots)$

$\mathrm{PA} + WP: \ (\varepsilon_0^2, \varepsilon_0 \cdot 2, \varepsilon_0, \varepsilon_0, \dots)$

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Reflection algebra

Reflection algebra of T is the structure

$$(\mathfrak{G}_T, \wedge_T, 1_T, \{R_n : n \in \omega\}).$$

Here, 1_T is the top element (the equivalence class of T).

We are interested in

- The identities of this structure;
- Its subalgebra generated by 1_T.

Reflection calculus RC

Language: $\alpha ::= \top \mid p \mid (\alpha \wedge \alpha) \mid n\alpha \qquad n \in \omega$
Example: $\alpha = 3(2p \wedge 32\top)$, or shortly: $3(2p \wedge 32)$.

Sequents: $\alpha \vdash \beta$.

RC rules:

1. $\alpha \vdash \alpha$; $\alpha \vdash \top$; if $\alpha \vdash \beta$ and $\beta \vdash \gamma$ then $\alpha \vdash \gamma$;
2. $\alpha \wedge \beta \vdash \alpha, \beta$; if $\alpha \vdash \beta$ and $\alpha \vdash \gamma$ then $\alpha \vdash \beta \wedge \gamma$;
3. $nn\alpha \vdash n\alpha$; if $\alpha \vdash \beta$ then $n\alpha \vdash n\beta$;
4. $n\alpha \vdash m\alpha$ for $n > m$;
5. $n\alpha \wedge m\beta \vdash n(\alpha \wedge m\beta)$ for $n > m$.

Ex. $3 \wedge 23 \vdash 3(\top \wedge 23) \vdash 323$.

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Main results on RC

Theorems (E. Dashkov, 2012).

1. $\alpha \vdash_{RC} \beta$ iff $\alpha \leq_T \beta$ holds in $(\mathfrak{G}_T; \wedge_T, 1_T, \{R_n : n \in \omega\})$;
2. RC is polytime decidable;
3. RC enjoys the finite model property.

Rem. The first claim is based on Japaridze's (1986) arithmetical completeness theorem for provability logic GLP.

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RC^0 as an ordinal notation system

Let RC^0 denote the variable-free fragment of RC.
Let W denote the set of all RC^0-formulas. For $\alpha, \beta \in W$ define:

- $\alpha \sim \beta$ if $\alpha \vdash \beta$ and $\beta \vdash \alpha$ in RC^0;
- $\alpha <_n \beta$ if $\beta \vdash n\alpha$.

Theorem.

1. Every $\alpha \in W$ is equivalent to a *word* (formula without $\wedge$);
2. $(W/{\sim}, <_0)$ is isomorphic to $(\varepsilon_0, <)$.

$\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$

Reduction property

Let $U \equiv_n V$ mean U and V prove the same Π_{n+1}-sentences.

Suppose $S \subseteq \Pi_{n+2}$ and $U \vdash S$.

Theorem. $R_{n+1}(U) \equiv_n R_n^\omega(U)$ in $\mathfrak{G}_S$.

Example. In $\mathfrak{G}_{\mathrm{EA}}$:

$I\Sigma_1 \equiv R_2(\mathrm{EA}) \equiv_1 R_1^\omega(\mathrm{EA}) \equiv \mathrm{PRA}$ (Parsons–Mints).

(This can be written as: $2 \equiv_1 \{1^k : k < \omega\}$ in $\mathfrak{G}_{\mathrm{EA}}$.)

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Explaining steps of the Worm game

Suppose $\alpha = (n+1)\beta \in W$, let α_S denote the value of α in $\mathfrak{G}_S$.

- $\alpha_S = R_{n+1}(U)$ where $U = S + \beta_S$.
- By the reduction property $R_{n+1}(U) \equiv_0 \{R_n^k(U) : k < \omega\}$.
- Iterations $R_n^k(U)$ correspond to the RC-formulas: $n\beta$, $n(\beta \wedge n\beta)$, $n(\beta \wedge n(\beta \wedge n\beta))\dots$
- Each of these formulas is RC-equivalent to a word.
- These words correspond to the main rule of the Worm sequence.

Iterated reflection and analysis of PA

- W_n is the set of words in the alphabet $\{k \in \omega : k \geq n\}$.
- We consider $(W_n, <_n)$ as an ordinal notation system and build the corresponding progression iterating R_n.
- Let S be a Π_{n+1} extension of EA.
  Each $\alpha \in W$ corresponds to an element $\alpha_S \in \mathfrak{G}_S$.

Theorem. For all $\alpha \in W_n$, theories α_S and $R_n^\alpha(S)$ over S are equivalent for Π_{n+1}-sentences.

Corollary. For each n, $\mathrm{PA} \equiv_n R_n^{\varepsilon_0}(\mathrm{EA})$ (U. Schmerl)

1. For $n = 0$: Consistency proof for PA (Gentzen);
2. For $n = 1$: Characterizing provably recursive functions of PA (Ackermann–Schwichtenberg–Wainer).

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- Lev D. Beklemishev, Fedor N. Pakhomov, *Reflection algebras and conservation results for theories of iterated truth*, 2019, 48 pp., arXiv: 1908.10302.