Proof Theory: From Arithmetic to Set theory

Michael Rathjen

University of Leeds

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Plan of the Talks

- **First Part**
  1. **From Hilbert to Gentzen.**
  2. **Gentzen's Hauptsatz and applications**
  3. **Consistency of PA**
- **Second Part:**
  1. **Ordinal representation systems**
  2. **Proof theory of (sub)systems of second order arithmetic and set theory**

The Origins of Proof Theory (Beweistheorie)

- Hilbert's second problem (1900): Consistency of Analysis
- Hilbert's Programme (1922,1925)

Paradoxes

- First paradox of set theory discovered by **Cantor** in 1895; communicated to Hilbert in 1896.
- Rediscovered by **Burali-Forti in 1897**.
- **Zermelo** (in Göttingen) discovered a paradox in set theory in 1900.
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- **Dedekind** 1888, 1890. Canonical requirement for a structural definition: Prove the existence of a system of things falling under the notion to ensure it does not contain **internal contradictions**.
- Hilbert 1904 (Heidelberg talk): Syntactic consistency proof for a weak system of arithmetic.
- Hilbert 1917 (Axiomatisches Denken): **we must turn the concept of a specifically mathematical proof itself into an object of investigation.**
- In 1917/18 Hilbert flirted again with logicism. Presented analysis in ramified type theory with the axiom of reducibility.
- Hilbert's finitist consistency program only emerged in the winter term 1921/22.

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Hilbert's Programme (1922,1925)

- **I. Codify the whole of mathematical reasoning in a formal theory T.**

- **II. Prove the consistency of T by finitistic means.**

- To carry out this task, Hilbert inaugurated a new mathematical discipline: **Beweistheorie** (**Proof Theory**).

- In Hilbert's Proof Theory, proofs become mathematical objects sui generis.

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Ackermann's Dissertation 1925

Consistency proof for a second-order version of **Primitive Recursive Arithmetic**.

Uses a finitistic version of **transfinite induction** up to the ordinal $\omega^{\omega^{\omega}}$.

Cantor's Representation of Ordinals

Theorem (Cantor, 1897) For every ordinal $\beta > 0$ there exist unique ordinals $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ such that

$$\beta = \omega^{\beta_0} + \ldots + \omega^{\beta_n}. \tag{1}$$

The representation of β in (1) is called the **Cantor normal form**.

We shall write $\beta =_{CNF} \omega^{\beta_1} + \cdots \omega^{\beta_n}$ to convey that $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_k$.

A Representation for ε_0

- ε_0 denotes the least ordinal $\alpha > 0$ such that
$$\beta < \alpha \;\Rightarrow\; \omega^\beta < \alpha.$$
- ε_0 is the least ordinal α such that $\omega^\alpha = \alpha$.
- $\beta < \varepsilon_0$ has a Cantor normal form with exponents $\beta_i < \beta$ and these exponents have Cantor normal forms with yet again smaller exponents. As this process must terminate, ordinals $< \varepsilon_0$ can be coded by natural numbers.

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Gentzen's Result

- **Gerhard Gentzen** showed that transfinite induction up to the ordinal

$$\varepsilon_0 \;=\; \mathsf{sup}\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\} \;=\; \mathsf{least}\ \alpha.\ \omega^\alpha = \alpha$$

suffices to prove the **consistency** of **Peano Arithmetic**, **PA**.

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Gentzen's Result in Detail

- $$\mathbf{F} + \text{PR-}\mathbf{TI}(\varepsilon_0) \vdash \mathbf{Con}(\mathbf{PA}),$$

  where **F** signifies a theory that is acceptable in finitism (e.g. $\mathbf{F} = \mathbf{PRA}$ = Primitive Recursive Arithmetic) and PR-**TI**(ε_0) stands for transfinite induction up to ε_0 for primitive recursive predicates.

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The Compelling Picture

The **non-finitist** part of **PA** is encapsulated in PR-**TI**(ε_0) and therefore "measured" by ε_0, thereby tempting one to adopt the following definition of **proof-theoretic ordinal** of a theory T:

$$|T|_{Con} \quad = \quad \text{least } \alpha.\ \mathbf{PRA} + \text{PR-}\mathbf{TI}(\alpha) \vdash \text{Con}(T).$$

Ordinally Informative Proof Theory

The two main strands of research are:

- **Cut Elimination** (and Proof Collapsing Techniques)
- Development of ever stronger **Ordinal Representation Systems**

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The Sequent Calculus

SEQUENTS

- A **sequent** is an expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, respectively.
- $\Gamma \Rightarrow \Delta$ is read, informally, as Γ yields Δ or, rather, the conjunction of the A_i yields the disjunction of the B_j.

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The Sequent Calculus

LOGICAL INFERENCES I

Negation

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ \neg\mathsf{L} \qquad \frac{B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg B}\ \neg\mathsf{R}$$

Implication

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Lambda \Rightarrow \Theta}{A \to B, \Gamma, \Lambda \Rightarrow \Delta, \Theta} \to \mathsf{L} \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \to \mathsf{R}$$

Conjunction

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \wedge \mathrm{L1} \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \wedge \mathrm{L2}$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \wedge \mathrm{R}$$

Disjunction

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \vee \mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \vee \mathrm{R1} \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} \vee \mathrm{R2}$$

The Sequent Calculus

LOGICAL INFERENCES II

Quantifiers

$$\frac{F(t), \Gamma \Rightarrow \Delta}{\forall x\, F(x), \Gamma \Rightarrow \Delta}\ \forall\, \mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, F(a)}{\Gamma \Rightarrow \Delta, \forall x\, F(x)}\ \forall\, \mathrm{R}$$

$$\frac{F(a), \Gamma \Rightarrow \Delta}{\exists x\, F(x), \Gamma \Rightarrow \Delta}\ \exists\, \mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, F(t)}{\Gamma \Rightarrow \Delta, \exists x\, F(x)}\ \exists\, \mathrm{R}$$

In $\forall$L and $\exists$R, t is an arbitrary term. The variable a in $\forall$R and $\exists$L is an eigenvariable of the respective inference, i.e. a is not to occur in the lower sequent.

The Sequent Calculus

AXIOMS

Identity Axiom

$$A \Rightarrow A$$

where A is any formula.

One could limit this axiom to the case of atomic formulae A

The Sequent Calculus

CUTS

CUT

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta}\ \text{Cut}$$

A is called the cut formula of the inference.

Example

$$\frac{B \Rightarrow A \qquad A \Rightarrow C}{B \Rightarrow C}\ \text{Cut}$$

The Sequent Calculus

STRUCTURAL RULES

Structural Rules — Exchange, Weakening, Contraction

$$\frac{\Gamma, A, B, \Lambda \Rightarrow \Delta}{\Gamma, B, A, \Lambda \Rightarrow \Delta}\ \mathcal{X}_l \qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}\ \mathcal{X}_r$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathcal{W}_l \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \mathcal{W}_r$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathcal{C}_l \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathcal{C}_r$$

The INTUITIONISTIC case

The intuitionistic sequent calculus is obtained by requiring that all sequents be intuitionistic.

A sequent $\Gamma \Rightarrow \Delta$ is said to be **intuitionistic** if Δ consists of at most one formula.

Specifically, in the intuitionistic sequent calculus there are no inferences corresponding to contraction right or exchange right.

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Classical Example

Our first example is a deduction of the law of excluded middle.

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, \neg A}\,\neg\mathrm{R}}{\Rightarrow A,\ A \vee \neg A}\,\vee\mathrm{R}}{\Rightarrow A \vee \neg A,\ A}\,\mathcal{X}_r}{\Rightarrow A \vee \neg A,\ A \vee \neg A}\,\vee\mathrm{R}}{\Rightarrow A \vee \neg A}\,\mathcal{C}_r
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Notice that the above proof is not intuitionistic since it involves sequents that are not intuitionistic.

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Intuitionistic Example

The second example is an intuitionistic deduction.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{F(a) \Rightarrow F(a)}{F(a) \Rightarrow \exists x\, F(x)}\ \exists R}{\neg\exists x\, F(x), F(a) \Rightarrow}\ \neg L}{F(a), \neg\exists x\, F(x) \Rightarrow}\ \mathcal{X}_I}{\neg\exists x F(x) \Rightarrow \neg F(a)}\ \neg L}{\neg\exists x\, F(x) \Rightarrow \forall x\, \neg F(x)}\ \forall R}{\Rightarrow \neg\exists x\, F(x) \to \forall x\, \neg F(x)}\ {\to}R$$

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Gentzen's Hauptsatz (1934)

Cut Elimination

If a sequent

$$\Gamma \Rightarrow \Delta$$

is provable, then it is provable without cuts.

Cut Elimination

EXAMPLE

Here is an example of how to eliminate cuts of a special form:

$$\dfrac{\dfrac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\,{\to}\mathrm{R} \qquad \dfrac{\Lambda \Rightarrow \Theta, A \qquad B, \Xi \Rightarrow \Phi}{A \to B, \Lambda, \Xi \Rightarrow \Theta, \Phi}\,{\to}\mathrm{L}}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi}\,\mathrm{Cut}$$

is replaced by

$$\dfrac{\dfrac{\Lambda \Rightarrow \Theta, A \qquad A, \Gamma \Rightarrow \Delta, B}{\Lambda, \Gamma \Rightarrow \Theta, \Delta, B}\,\mathrm{Cut} \qquad B, \Xi \Rightarrow \Phi}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi}\,\mathrm{Cut}$$

The Subformula Property

The Hauptsatz has an important consequence:

SUBFORMULA PROPERTY

If a sequent $\Gamma \Rightarrow \Delta$ is provable, then it has a deduction all of whose formulae are subformulae of the formulae in Γ and Δ.

COROLLARY

A contradiction, i.e. the empty sequent, is not deducible.

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Applications of the Haupsatz

- Herbrand's Theorem in LK (classical):

$$\vdash \exists x R(x) \quad \text{implies} \quad \vdash R(t_1) \vee \ldots \vee R(t_n)$$

  some t_i (R quantifier-free).
- Extended Herbrand's Theorem in LK:

$$\vdash \Gamma \Rightarrow \exists x R(x) \quad \text{implies} \quad \vdash \Gamma \Rightarrow R(t_1) \vee \ldots \vee R(t_n)$$

  some t_i (R quantifier-free, Γ purely universal).
- In **LJ** (intuitionistic predicate logic):

$$\vdash \Gamma \Rightarrow \exists x R(x) \quad \text{implies} \quad \vdash \Gamma \Rightarrow R(t)$$

  for some term t where Γ is $\vee$ and $\exists$ free.
- Hilbert-Ackermann Consistency
- If T is a **geometric theory** and T classically proves a **geometric implication** A then T intuitionistically proves A.

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Theories and Cut Elimination

- What happens when we try to apply the procedure of cut elimination to theories?
- Axioms are detrimental to this procedure. It breaks down because the symmetry of the sequent calculus is lost. In general, we cannot remove cuts from deductions in a theory T when the cut formula is an axiom of T.
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- Gives rise to

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- This is a very important tool in proof theory. For example, it works very well if the axioms of a theory can be presented as atomic intuitionistic sequents (also called Horn clauses), yielding the completeness of Robinsons **resolution method**.

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Partial cut elimination also pays off in the case of fragments of **PA** and set theory with restricted induction schemes, be it induction on natural numbers or sets. This method can be used to extract bounds from proofs of Π^0_2 statements in such fragments.

Gentzen's way out

- Gentzen defined an assignment ord of ordinals to derivations of **PA** such for every derivation D of **PA** in his sequent calculus,

$$\mathrm{ord}(D) < \varepsilon_0.$$

- He then defined a reduction procedure $\mathcal{R}$ such that whenever D is a derivation of the empty sequent in **PA** then $\mathcal{R}(D)$ is another derivation of the empty sequent in **PA** but with a smaller ordinal assigned to it, i.e.,

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- If $\mathrm{PRWO}(\varepsilon_0)$ is the statement that there are no infinitely descending primitive recursive sequences of ordinals below ε_0, then the following are immediate consequences of Gentzen's work.

Theorem: (Gentzen 1936, 1938)

(i) *The theory of primitive recursive arithmetic,* **PRA**, *proves that* $\mathrm{PRWO}(\varepsilon_0)$ *implies the 1-consistency of* **PA**.

(ii) *Assuming that* **PA** *is consistent,* **PA** *does not prove* $\mathrm{PRWO}(\varepsilon_0)$.

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The ω-rule

- Infinite proofs were considered by Brouwer (1927) and Zermelo (1932, 1935) to be the “right” proofs.
- Hilbert 1930/31: Die Grundlegung der elementaren Zahlenlehre, with $F(x)$ primitive recursive:

$$\frac{\Gamma \Rightarrow \Delta, F(0);\ \Gamma \Rightarrow \Delta, F(1);\ \ldots\ ; \Gamma \Rightarrow \Delta, F(n);\ \ldots}{\Gamma \Rightarrow \Delta, \forall x\, F(x)}\ \omega\mathrm{R}$$

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Partial cut elimination also pays off in the case of fragments of **PA** and set theory with restricted induction schemes, be it induction on natural numbers or sets. This method can be used to extract bounds from proofs of Π^0_2 statements in such fragments.

Full arithmetic, i.e. **PA**, does not even allow for partial cut elimination since the induction axioms have unbounded complexity.

However, one can remove the obstacle against cut elimination in a drastic way by going **infinite**. The so-called ω-**rule** consists of the two types of infinitary inferences:

$$\frac{\Gamma \Rightarrow \Delta, F(0);\ \Gamma \Rightarrow \Delta, F(1);\ \ldots\ ; \Gamma \Rightarrow \Delta, F(n);\ \ldots}{\Gamma \Rightarrow \Delta, \forall x\, F(x)}\ \omega\mathrm{R}$$

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The price to pay will be that deductions become infinite objects, i.e. infinite well-founded trees.

The sequent-style version of Peano arithmetic with the ω-rule will be termed $\mathbf{PA}_\omega$.

With the aid of the ω-rule, the induction schema becomes logically deducible in infinitary logic.

Cut Elimination for $\mathbf{PA}_\omega$

We want to measure the **height** and **cut rank** of a $\mathbf{PA}_\omega$ deduction $\mathcal{D}$. We will notate this by

$$\mathcal{D} \mid\!\frac{\alpha}{k}\, \Gamma \Rightarrow \Delta\,.$$

Embedding Theorem If $\mathbf{PA} \vdash \Gamma \Rightarrow \Delta$ then

$$\mathbf{PA}_\omega \; \frac{|\omega+m}{|k} \; \Gamma \Rightarrow \Delta$$

for some $m, k < \omega$.

Reduction Lemma If $\mathbf{PA}_\omega \mid\!\frac{\alpha}{k}\, \Gamma \Rightarrow \Delta, A$ and $\mathbf{PA}_\omega \mid\!\frac{\beta}{k}\, A, \Lambda \Rightarrow \Theta$ with $k = |A|$, then

$$\mathbf{PA}_\omega \mid\!\frac{\alpha\#\beta}{k}\, \Gamma, \Lambda \Rightarrow \Delta, \Theta\,.$$

Theorem If $\mathbf{PA}_\omega \stackrel{\alpha}{\underset{k+1}{\vdash}} \Gamma \Rightarrow \Delta$, then $\mathbf{PA}_\omega \stackrel{\omega^\alpha}{\underset{k}{\vdash}} \Gamma \Rightarrow \Delta$.

Cut Elimination Theorem If $\mathbf{PA}_\omega \stackrel{\alpha}{\underset{n}{\vdash}} \Gamma \Rightarrow \Delta$, then

$$\mathbf{PA}_\omega \stackrel{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^\alpha}}}}}}{\underset{0}{\vdash}} \Gamma \Rightarrow \Delta \qquad \underbrace{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^\alpha}}}}}}_{n \text{ times}}.$$

The Finite Order Sequent Calculus, **GLC**

Quantifiers

$$\frac{F(\{v \mid A(v)\}), \Gamma \Rightarrow \Delta}{\forall X\, F(X), \Gamma \Rightarrow \Delta}\ \forall_2\,\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, F(U)}{\Gamma \Rightarrow \Delta, \forall X\, F(X)}\ \forall_2\,\mathrm{R}$$

$$\frac{F(U), \Gamma \Rightarrow \Delta}{\exists X\, F(X), \Gamma \Rightarrow \Delta}\ \exists_2\,\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, F(\{v \mid A(v)\})}{\Gamma \Rightarrow \Delta, \exists X\, F(X)}\ \exists_2\,\mathrm{R}$$

In $\forall_2$L and $\exists_2$R, $A(a)$ is an arbitrary formula. The variable U in $\forall_2$R and $\exists_2$L is an eigenvariable of the respective inference, i.e. U is not to occur in the lower sequent.

Birth of Second Order Proof Theory by The Fundamental Conjecture on **GLC**

The Fundamental Conjecture **FC** for **GLC** asserts that the Hauptsatz holds for **GLC**.

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Second order arithmetic; $\mathbf{Z_2}$ aka Analysis

$\mathbf{Z_2}$ is a two sorted formal system. Extends **PA**.

- Variables $n, m, \ldots$ range over natural numbers.
  Variables $X, Y, Z, \ldots$ range over sets of natural numbers.
  Relation symbols $=, <, \in$. Function symbols $+, \times, \ldots$

- Comprehension Principle/Axiom:

  For any property P definable in the language of $\mathbf{Z_2}$,

  $$\{n \in \mathbb{N} \mid P(n)\}$$

  is a set; or more formally

  $$\text{(CA)} \qquad \exists X \, \forall n \, [n \in X \leftrightarrow A(x)]$$

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Stratification of Comprehension

- A Π^1_k-formula (Σ^1_k-formula) is a formula of $\mathbf{Z_2}$ of the form

  $$\forall X_1 \dots QX_k\, A(X_1, \dots, X_k) \qquad (\exists X_1 \dots QX_k\, A(X_1, \dots, X_k))$$

  with $\forall X_1 \dots QX_k$ ($\exists X_1 \dots QX_k$) a string of k alternating set quantifiers, beginning with a universal quantifier (existential quantifier), followed by a formula $A(X_1, \dots, X_k)$ without set quantifiers.

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Subsystems of $\mathbf{Z_2}$

- Basic arithmetical axioms in all subtheories of $\mathbf{Z_2}$ are: defining axioms for $0, 1, +, \times, E, <$ (as for **PA**) and the *induction axiom*

$$\forall X\,[0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X) \rightarrow \forall n\,(n \in X)].$$

- For each axiom scheme **Ax**, $(\mathbf{Ax})_0$ denotes the theory consisting of the basic arithmetical axioms plus the scheme **Ax**.
- $(\mathbf{Ax})$ stands for the theory $(\mathbf{Ax})_0$ augmented by the scheme of induction for all $\mathcal{L}_2$-formulae.
- Let $\mathcal{F}$ be a collection of formulae of $\mathbf{Z_2}$.
  Another important axiom scheme for formulae F in $\mathcal{C}$ is

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For many mathematical theorems τ, there is a weakest natural subsystem $S(\tau)$ of $\mathbf{Z_2}$ such that $S(\tau)$ proves τ.
Moreover, it has turned out that $S(\tau)$ often belongs to a small list of specific subsystems of $\mathbf{Z_2}$. Reverse Mathematics has singled out five subsystems of $\mathbf{Z_2}$:

- $\mathbf{RCA}_0$ Recursive Comprehension
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Mathematical Equivalences: Examples

- **RCA**$_0$ "Every countable field has an algebraic closure"; "Every countable ordered field has a real closure"
- **WKL**$_0$ "Cauchy-Peano existence theorem for solutions of ordinary differential equations"; "Hahn-Banch theorem for separable Banach spaces"
- **ACA**$_0$ "Bolzano-Weierstrass theorem"; "Every countable commutative ring with a unit has a maximal ideal"
- **ATR**$_0$ "Every countable reduced abelian p-group has an Ulm resolution"
- $(\Pi^1_1-\mathbf{CA})_0$ "Every uncountable closed set of real numbers is the union of a perfect set and a countable set"; "Every countable abelian group is a direct sum of a divisible group and a reduced group"

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A brief history of ordinal representation systems

1904-1950

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Veblen extended the initial segment of the countable ordinals for which unique representations can be given effectively.

- The **derivative** f' of a function $f : \mathrm{ON} \to \mathrm{ON}$ is the function which enumerates in increasing order the solutions of the equation

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- $\varphi_0(\xi) \ = \ \omega^{\xi}$
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Veblen Normal Form

Theorem. For every ordinal $\alpha > 0$ there exist uniquely determined ordinals $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ such that:

- $$\alpha = \varphi_{\xi_1}(\eta_1) + \ldots + \varphi_{\xi_n}(\eta_n)$$
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only a finite number of the arguments

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- Veblen singled out the ordinal $E(0)$, where $E(0)$ is the least ordinal $\delta > 0$ which cannot be named in terms of functions

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- Bachmann's novel idea: Use **uncountable ordinals** to keep track of the functions defined by diagonalization.
- Define a set of ordinals $\mathfrak{B}$ closed under successor such that with each limit $\lambda \in \mathfrak{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathfrak{B}$ of length $\tau_\lambda \leq \mathfrak{B}$ and $\lim_{\xi < \tau_\lambda} \lambda[\xi] = \lambda$.
- Let Ω be the first uncountable ordinal. A hierarchy of functions $(\overset{\mathfrak{B}}{\varphi}_\alpha)_{\alpha \in \mathfrak{B}}$ is then obtained as follows:

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- Kripke-Platek set theory, **KP**, is a fragment of **ZFC**.
- **KP** has no Choice, no Power Set Axiom
- Separation and Collection are restricted to formulas with bounded quantifiers
- Admissible Sets are transitive models of **KP**
- Admissible Ordinals: ordinals α satisfying $\mathbf{L}_\alpha \models \mathbf{KP}$
- Gödel's Constructible Hierarchy **L**:

$$\begin{aligned} \mathbf{L}_0 &= \emptyset, \\ \mathbf{L}_\lambda &= \bigcup\{\mathbf{L}_\beta : \beta < \lambda\} \ \lambda \text{ limit} \\ \mathbf{L}_{\beta+1} &= \big\{X : X \subseteq \mathbf{L}_\beta;\ X \text{ definable over } \langle \mathbf{L}_\beta, \in\rangle\big\}. \end{aligned}$$

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The **axioms** of **KP** are:

Extensionality:	$a = b \rightarrow [F(a) \leftrightarrow F(b)]$
Foundation:	$\exists x G(x) \rightarrow \exists x[G(x) \wedge (\forall y \in x)\neg G(y)]$
Pair:	$\exists x\,(x = \{a, b\})$.
Union:	$\exists x\,(x = \bigcup a)$.
Infinity:	$\exists x\,\big[x \neq \emptyset\ \wedge\ (\forall y \in x)(\exists z \in x)(y \in z)\big]$.
Δ_0 Separation:	$\exists x\,\big(x = \{y \in a : F(y)\}\big)$
	$F(y)$ Δ_0-formula.
Δ_0 Collection:	$(\forall x \in a)\exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z) G(x, y)$
	for all Δ_0–formulas G.

By a Δ_0 formula we mean a formula of set theory in which all the quantifiers appear restricted, that is have one of the forms $(\forall x \in b)$ or $(\exists x \in b)$.

Ordinal analysis of **KP**

Bachmann's ordinal representation system, $\mathbb{B}$, can be used for an ordinal analysis of **KP**.

This was done by Gerhard Jäger in the late 1970s.

Build a formal constructible hierarchy of terms along the ordinals of $\mathbb{B}$, including terms $\mathbb{L}_\alpha$.

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  $(\Delta^1_2\text{-}\mathbf{CA})$
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- Buchholz, Pohlers, Sieg 1977
  Theories of Iterated Inductive Definitions
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  $\Omega_{\nu+1}$-rules
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  $(\Sigma^1_2\text{-}\mathbf{AC}) + \mathbf{BI}$, **KPi**
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  First-order reflection
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$$|\mathbf{ACA}_0| = \varepsilon_0$$

$$|\mathbf{RCA}_0| = \omega^\omega = |\mathbf{WKL}_0|$$

$$0$$

$$|(\Sigma^1_2\text{-}\mathbf{AC}) + \mathbf{BI}| = \psi_{\Omega_1} I$$

$$|(\Delta^1_2\text{-}\mathbf{CA})| = \psi_{\Omega_1}\Omega_{\varepsilon_0}$$

$$|(\Pi^1_1-\mathbf{CA})_0| = \psi_{\Omega_1}\Omega_\omega$$

$$|\mathbf{ATR}_0| = \Gamma_0$$

$$|(\Sigma^1_2\text{-}\mathbf{AC}) + \mathbf{BI}| = \psi_{\Omega_1} I$$

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Π^1_3-comprehension is the current barrier.

Is this the generic case?

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谢谢！