Well ordering principles and a uniform Kruskal theorem

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The ordinal ε_0

Recall the ordinal number / well order $\varepsilon_0 = \min\{\beta \,|\, \omega^\beta = \beta\}$.

Ordinals below ε_0 can be represented by the terms generated by

$$\alpha \quad ::= \quad 0 \quad | \quad \omega^{\alpha_0} + \ldots + \omega^{\alpha_n} \quad (\alpha_0 \geqslant \ldots \geqslant \alpha_n),$$

which can be seen as formal Cantor normal forms. Here $\leqslant$ is the lexicographic order on exponents (with recursive calls).

Example: We have

$$[\,\omega + 2 = \,]\; \omega^{\omega^0} + \omega^0 + \omega^0 \leqslant \omega^{\omega^0+\omega^0}\; [\, = \omega^2\,]$$

because of $\omega^0 < \omega^0 + \omega^0$.

Unprovability of well foundedness

Theorem (G. Gentzen 1936): Peano arithmetic (PA) cannot prove that ε_0 is well founded (in the sense that no primitive recursive sequence of ordinal terms descends forever).

Proof: By Gödel's theorem, it suffices to show that the well foundedness of ε_0 entails the consistency of PA, provably in PA.

Label proofs by ordinals $< \varepsilon_0$, such that each (purported) proof of a contradiction can be transformed into one with a smaller label.

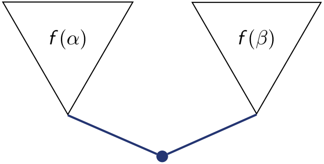
Now if there was a proof of contradiction, we would get an infinite sequence of proofs with descending ordinal labels. $\square$

A mathematical incompleteness

Corollary (D. de Jongh, D. Schmidt, H. Friedman ∼1980):
Peano arithmetic cannot prove the binary Kruskal theorem: For each (primitive recursive) sequence $t_0, t_1, \ldots$ of finite binary trees, there are $i < j$ such that t_i embeds into t_j.

Proof: Define a primitive recursive function $f : \varepsilon_0 \to$ "binary trees" such that $\alpha \leqslant \beta$ holds if $f(\alpha)$ embeds into $f(\beta)$:

$$f(\omega^\alpha + \beta) =$$



Second order arithmetic

In second order arithmetic one can quantify over natural numbers (finite objects) and subsets of $\mathbb{N}$ (countable collections).

Arithmetical comprehension is the principle

$$\exists X \subseteq \mathbb{N}\, \forall n \in \mathbb{N}\, (n \in X \leftrightarrow \varphi(n)),$$

for φ without set quantifiers (but possibly with set parameters).

Arithmetical recursion along $\mathbb{N}$ allows to form sequences $(X_k)_{k \in \mathbb{N}}$ of sets $X_k \subseteq \mathbb{N}$ (coded by a single set $\{\langle k, n\rangle \,|\, n \in X_k\}$) with

$$X_{n+1} = \{n \in \mathbb{N} \,|\, \varphi(n, X_n)\},$$

for φ as above.

ω-models

In an ω-model of second order arithmetic, number quantifiers range over the **standard model $\mathbb{N}$**, while set quantifiers range over some **subset $\mathcal{M} \subseteq \mathcal{P}(\mathbb{N})$ of the power set**.

Intuitively, $\mathcal{M}$ tells us which (closure properties of) infinite sets we need to make a statement true.

Observation: The statement that ε_0 is well founded holds in all ω-models, since it does not assert the existence of infinite sets.

Arithmetical comprehension does not hold in all ω-models, since it asserts, e. g., the existence of Turing's halting set.

Order transformations and set existence

Let $\gamma \mapsto \varepsilon_\gamma$ enumerate the (club) set $\{\beta \,|\, \omega^\beta = \beta\}$ of ordinals.

Ordinals below ε_γ can be represented by the terms generated by

$$\alpha \quad ::= \quad 0 \quad | \quad \varepsilon_\beta \quad (\beta < \gamma) \quad | \quad \omega^{\alpha_0} + \ldots + \omega^{\alpha_n} \quad (\alpha_0 \geqslant \ldots \geqslant \alpha_n).$$

Here ε_β is a constant, indexed by the ordinal (not term) $\beta < \gamma$.

> **Observation:** The statement that $\gamma \mapsto \varepsilon_\gamma$ preserves well foundedness asserts the existence of infinite sets, since it says: for any descending sequence in ε_γ, **there is** a descending sequence in γ.

An equivalence

Theorem (Marcone & Montalbán 2007, Afshari & Rathjen 2009):
The following are equivalent (provably in RCA_0):

(i) $\gamma \mapsto \varepsilon_\gamma$ preserves well foundedness,

(ii) arithmetical recursion along $\mathbb{N}$,

(iii) every subset of $\mathbb{N}$ is contained in a (countable coded) ω-model of arithmetical comprehension (and the latter holds).

We point out that the cited references give rather **different proofs via computability theory and proof theory**, respectively.

From well foundedness to ω-model

An ω-proof is a well founded deduction tree using the ω-rule

$$\frac{\varphi(\overline{0}) \vee \Gamma \quad \varphi(\overline{1}) \vee \Gamma \quad \cdots \quad \varphi(\overline{n}) \vee \Gamma \quad \cdots}{\forall n \in \mathbb{N}\, \varphi(n) \vee \Gamma}.$$

By the **Shoenfield completeness theorem**, we get an ω-model of ACA_0 if there is no ω-proof of a contradiction from ACA_0.

To show that there is no ω-proof of contradiction, one relativizes Gentzen's ordinal analysis to a given ω-proof. Here $\gamma \mapsto \varepsilon_\gamma$ plays the role that was previously played by ε_0.

A mathematical equivalence

We call X a **well partial order** (WPO) if the following holds: for any infinite sequence $x_0, x_1, \ldots \subseteq X$ there are $i < j$ with $x_i \leqslant_X x_j$.

Let $\mathcal{B}(X)$ be the set of binary trees with leaves labelled by elements of X. The previous theorem and a result of D. Schmidt yield:

Corollary: The following are equivalent over RCA_0:

(i) If X is a WPO, then so is the embeddability relation on $\mathcal{B}(X)$, i. e., Kruskal's theorem holds for $\mathcal{B}(X)$.

(ii) Arithmetical recursion along $\mathbb{N}$ holds.

A wealth of results . . .

A **well ordering principle** (of type one) is a statement of the form "$\gamma \mapsto D(\gamma)$ preserves well foundedness" for computable D.

Many important principles of reverse mathematics are equivalent to well ordering principles with natural D:

- arithmetical comprehension (Girard, Hirst),
- arithmetical transfinite recursion
  (Friedman, Rathjen & Weiermann, Marcone & Montalbán),
- ω-models of bar induction (Rathjen & Valencia-Vizcaíno), . . .

. . . and a limitation

The principle of Π^1_1-comprehension asserts

$$\exists X \subseteq \mathbb{N}\, \forall n \in \mathbb{N}\, (n \in X \leftrightarrow \forall Y \subseteq \mathbb{N}\, \varphi_0(n, Y)),$$

for formulas φ_0 without set quantifiers.

The principles "$\gamma \mapsto D(\gamma)$ preserves well foundedness" are Π^1_2 (of the form $\forall X \subseteq \mathbb{N}\, \exists Y \subseteq \mathbb{N}\, \varphi_0$). It is known that Π^1_1-comprehension cannot be equivalent to statements of this form.

To overcome this limitation, we will consider well ordering principles of **higher type**, i. e., transformations $D \mapsto \mathcal{F}(D)$ with D as above.

A class of uniform transformations

Let ON be the category of ordinals and strictly increasing functions.

Definition (essentially Girard ∼1980): A dilator consists of a functor $D : \mathrm{ON} \to \mathrm{ON}$ and a natural family of functions

$$\mathrm{supp}_\beta : D(\beta) \to \text{“finite subsets of } \beta\text{”}$$

that satisfy the following: If the support of $\sigma \in D(\beta)$ is contained in the range of $f : \alpha \to \beta$ (i. e. if $\mathrm{supp}_\beta(\sigma) \subseteq \mathrm{rng}(f)$), then σ lies in the range of $D(f) : D(\alpha) \to D(\beta)$ (i. e. we have $\sigma \in \mathrm{rng}(D(f))$).

Fact: Dilators are determined by their restrictions to finite ordinals (and morphisms), which are objects of second order arithmetic.

An example

$$D(\gamma) = 1 + \gamma^2 \cong \{0\} \cup \{\langle \alpha, \beta \rangle \,|\, \alpha, \beta < \gamma\},$$

$$D(f)(0) = 0,$$
$$D(f)(\langle \alpha, \beta \rangle) = \langle f(\alpha), f(\beta) \rangle,$$

$$\mathrm{supp}_\gamma(0) = \emptyset,$$
$$\mathrm{supp}_\gamma(\langle \alpha, \beta \rangle) = \{\alpha, \beta\}.$$

Bachmann-Howard fixed points

Definition (F.): A Bachmann-Howard fixed point of a dilator D consists of an ordinal γ and a function $\vartheta : D(\gamma) \to \gamma$ such that

(i) $\sigma < \tau \in D(\gamma)$ implies $\vartheta(\sigma) < \vartheta(\tau)$, under the side condition that $\alpha < \vartheta(\tau)$ holds for all $\alpha \in \operatorname{supp}_\gamma(\sigma)$,

(ii) we have $\alpha < \vartheta(\sigma)$ for any $\sigma \in D(\gamma)$ and $\alpha \in \operatorname{supp}_\gamma(\sigma)$.

Remark: The definition is inspired by the Bachmann-Howard ordinal (specifically by Rathjen's notation system).

The side condition in (i) cannot be avoided, since $D(\gamma) > \gamma$ could hold for all γ, so that $\vartheta : D(\gamma) \to \gamma$ cannot be strictly increasing.

Example, continued

Once again, consider the dilator with

$$D(\gamma) = 1 + \gamma^2 \cong \{0\} \cup \{\langle\alpha,\beta\rangle \,|\, \alpha,\beta < \gamma\},$$
$$\mathsf{supp}_\gamma(0) = \emptyset, \quad \mathsf{supp}_\gamma(\langle\alpha,\beta\rangle) = \{\alpha,\beta\}.$$

A Bachmann-Howard fixed point is given by $\vartheta : D(\varepsilon_0) \to \varepsilon_0$ with

$$\vartheta(0) = 0, \quad \vartheta(\langle\alpha,\beta\rangle) = \omega^{\omega^\alpha} \cdot (\beta + 1).$$

Conversely, if $\vartheta : D(\gamma) \to \gamma$ is a Bachmann-Howard fixed point, then we get an embedding $f : \varepsilon_0 \to \gamma$ by setting

$$f(0) = 0, \quad f(\omega^\alpha + \beta) = \vartheta(\langle\alpha,\beta\rangle).$$

A higher well ordering principle

Each dilator D has an initial Bachmann-Howard fixed point ϑD, and the transformation $D \mapsto \vartheta D$ is computable.

Theorem (F. 2019): The following are equivalent over RCA_0:

(i) if D is a dilator, then ϑD is well founded,

(ii) Π^1_1-comprehension.

The proof uses G. Jäger's result that (ii) is equivalent to the following statement (over a weak set theory):

(iii) every set lies in a transitive model of Kripke-Platek set theory.

Constructing transitive models (1/2)

Consider Gödel's constructible hierarchy $\mathbb{L}$. Roughly speaking, an $\mathbb{L}_\alpha$-proof is a well founded deduction tree using the rule

$$\frac{\cdots \quad \varphi(c) \vee \Gamma \quad \cdots \qquad (\text{all } c \in \mathbb{L}_\alpha)}{\forall x\, \varphi(x) \vee \Gamma}.$$

Completeness (F.): If there is no $\mathbb{L}_\alpha$-proof of contradiction in Kripke-Platek set theory (KP), there is a $\beta < \alpha$ with $\mathbb{L}_\beta \vDash \mathrm{KP}$.

To get a transitive model of KP, it suffices to show that there cannot be $\mathbb{L}_\alpha$-proofs of contradiction for all $\alpha \in \mathrm{ON}$.

Constructing transitive models (2/2)

Aiming at a contradiction, assume that P_α is an $\mathbb{L}_\alpha$-proof of contradiction in Kripke-Platek set theory, for every $\alpha \in \mathrm{ON}$.

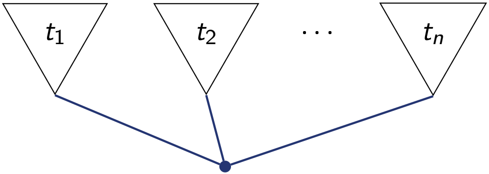
One can arrange that the constructions $\alpha \mapsto \mathbb{L}_\alpha$ and $\alpha \mapsto P_\alpha$ are functorial, to get a dilator D such that $D(\alpha)$ bounds the rank of P_α.

As in the desired theorem, assume that the Bachmann-Howard fixed point ϑD is well founded, i. e., an ordinal number.

Using the collapse $\vartheta : D(\vartheta D) \to \vartheta D$, one can emulate Jäger's ordinal analysis of Kripke-Platek set theory, to show that $P_{\vartheta D}$ cannot prove a contradiction after all.

Finite trees, once again

The collection of finite trees is the **minimal fixed point** of the transformation $X \mapsto$ "finite sequences with entries in X":



Similarly, one obtains finite sequences with entries in Y (as the fixed point of $X \mapsto 1 + X \times Y$) and other **recursive data types**.

Uniform transformations of partial orders

Let WPO be the category of well partial orders and order reflections (i. e., maps $f : X \to Y$ such that $f(x) \leqslant_Y f(x')$ implies $x \leqslant_X x'$).

Definition: A **WPO-dilator** consists of

(i) a functor $W : \mathrm{WPO} \to \mathrm{WPO}$ that preserves embeddings,

(ii) natural functions $\mathrm{supp}_X : W(X) \to$ "finite subsets of X" with

$$\mathrm{supp}_Y(\sigma) \subseteq \mathrm{rng}(f) \quad \Rightarrow \quad \sigma \in \mathrm{rng}(W(f))$$

for any embedding $f : X \to Y$ and any $\sigma \in W(Y)$.

We say that W is **normal** if we have

$$\sigma \leqslant_{W(X)} \tau \quad \Rightarrow \quad \forall x \in \mathrm{supp}_X(\sigma) \exists x' \in \mathrm{supp}_X(\tau) : x \leqslant_X x'.$$

Kruskal fixed points

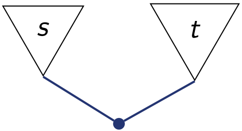
Definition: A Kruskal fixed point of a normal WPO-dilator W consists of a well partial order X and a map $\kappa : W(X) \to X$ with

$$\kappa(\sigma) \leqslant_X \kappa(\tau) \quad \Leftrightarrow \quad \sigma \leqslant_{W(X)} \tau \text{ or } \exists x \in \operatorname{supp}_X(\tau) : \kappa(\sigma) \leqslant_X x.$$

Example: Let $W(X) = 1 + X \times X$ with $\operatorname{supp}_X(\langle x, y\rangle) = \{x, y\}$ and

$$\langle x, y\rangle \leqslant_{W(X)} \langle x', y'\rangle \quad \Leftrightarrow \quad x \leqslant_X x' \text{ and } y \leqslant_X y'.$$

A Kruskal fixed point is given by the set of binary trees, with

$\kappa(\langle s, t\rangle) =$ 

A uniform Kruskal theorem

Each normal WPO-dilator W has an initial Kruskal fixed point $\mathcal{T}W$, and the transformation $W \mapsto \mathcal{T}W$ is computable.

Theorem (F. & Rathjen & Weiermann 2020): The following are equivalent, over RCA_0 plus the chain antichain principle:

(i) Π^1_1-comprehension,

(ii) the **uniform Kruskal theorem**: if W is a normal WPO-dilator, then $\mathcal{T}W$ is a well partial order.

Furthermore, A. Marcone has shown that (i) is equivalent to

(iii) the "minimal bad sequence"-lemma of Nash-Williams.

The proof idea

To obtain Π^1_1-comprehension, it suffices to show that ϑD is well founded for an arbitrary dilator D, by the theorem before.

The crucial step is to construct a normal WPO-dilator W_D that admits a natural family of order reflecting maps

$$\eta_X : D(X) \to W_D(X) \qquad \text{(for each linear order } X\text{).}$$

One can then construct an order reflection $f : \vartheta D \Rightarrow \mathcal{T} W_D$.
By the uniform Kruskal theorem, $\mathcal{T} W_D$ is a well partial order.
It follows that ϑD is well founded, as desired.

Conclusion and Thanks

By considering order transformations we can

- connect proof theory, reverse mathematics and combinatorics,
- complement traditional ordinal analysis by more abstract and very elegant results (e. g. on Bachmann-Howard fixed points).

Details and further references can be found in

- A. Freund, Π^1_1*-comprehension as a well-ordering principle*, Advances in Mathematics 355 (2019), 106767, 65 pp.
- A. Freund, M. Rathjen and A. Weiermann, *Minimal bad sequences are necessary for a uniform Kruskal theorem*, preprint available as `arXiv:2001.06380`.