Proof Theory: From the Foundations of Mathematics to Applications in Core Mathematics

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Early history

Hilbert's Program:

Establish that uses of higher noneffective/transfinite **(,,ideal") principles** $\mathcal{I}$ in proofs of combinatorial/finitistic **(,,real") propositions** $\mathcal{P}$ can be **eliminated**, at least in principle.

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Though in this specific form, in general impossible (Gödel), the basic approach is largely correct for existing ordinary mathematics: **proof-theoretic tameness** of ordinary mathematics!

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- If

$$\mathbf{(0 = 1)^I \equiv (0 = 1)}$$

  this reduces the consistency problem of $\mathcal{T}_1$ to that of $\mathcal{T}_2$.

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Uses embedding into systems based in **intuitionistic logic** (**Brouwer**).

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- $A \leftrightarrow A^G$ by **quantifier-free choice** in all types

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- $\underline{x}, \underline{y}$ are tuples of **functionals of finite type** over $\mathbb{N}, X_1, \ldots, X_n$ (X_i abstract metric or normed spaces).
- Functional interpretations **extracts uniform bounds** on A^G from a given **proof** of A.

Proof Mining in core mathematics

- During (mainly) the last 20 years this proof-theoretic approach has resulted in **numerous new quantitative results** as well as **qualitative uniformity results** in particular in: nonlinear analysis, fixed point theory, ergodic theory, topological dynamics, approximation theory, convex optimization, abstract Cauchy problems, pursuit-evasion games (≥ 100 papers mostly in specialized journals in the resp. areas or general mathematics journals).

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- General **logical metatheorems** explain applications as instances of logical phenomena (K. 2005, Gerhardy/K. 2008, TAMS).
- Some of the logical tools used have been rediscovered in 2007 in special cases by Terence Tao prompted by concrete mathematical needs **"finitary analysis"!**

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Examples of such spaces X : metric, normed, Hilbert, uniformly convex uniformly smooth, hyperbolic, CAT(0), abstract L^p and $C(K)$ spaces (but e.g. not separable, strictly convex or smooth spaces).

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- **Final product: explicit bound** with **ordinary analytical proof** for its correctness (no reference to logic!).

The running theme: convergence statements in analysis

Let (x_n) be a Cauchy sequence in a metric space (X, d), i.e.

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A bound $\Phi(k, g)$ on '$\exists n$' in the latter formula is a **rate of metastability**.

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- Usually **possible for asymptotic regularity** results

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- Possible also in the nonunique case for **Fejér monotone algorithms** if one has a **modulus of metric regularity** (see below).

Applications to Convex Optimization

A polynomial rate of asymptotic regularity in Bauschke's solution of the 'zero displacement conjecture'

Consider a Hilbert space H and nonempty closed and convex subsets $C_1, \ldots, C_N \subseteq H$ with metric projections P_{C_i}, define $T := P_{C_N} \circ \ldots \circ P_{C_1}$. In 2003 Bauschke proved the 'zero displacement conjecture':

$$\|T^{n+1}x - T^n x\| \to 0 \quad (x \in H).$$

Previously only known for $N = 2$ or $Fix(T) \neq \emptyset$ (or even $\bigcap_{i=1}^N C_i \neq \emptyset$) or C_i half spaces etc.

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Proof uses abstract theory of maximal monotone operators: Minty's theorem, Brézis-Haraux theorem, Rockafellar's maximal monotonicity and sum theorems, Bruck-Reich theory of strongly nonexpansive mappings, conjugate functions, normal cone operator...).

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Logical metatheorems, therefore, guarantee the extractability a uniform rate of asymptotic regularity which only depends on the error $\varepsilon > 0$, $N \in \mathbb{N}$ and **majorants** for $x \in H$ and $P_{C_1}, \ldots, P_{C_N}$:

$b \geq \|x\|$ and $K \geq \|c_1\|, \ldots, \|c_N\|$ for some **arbitrary** points $c_1 \in C_1, \ldots, c_N \in C_N$:

$$\|P_{C_i}0\| \leq \|c_i\| \leq K.$$

Since the mappings P_{C_i} are nonexpansive, the corollary guarantees a computable $\Phi(\varepsilon, N, b, K)$ s.t. for $b \geq \|x\|$

$$\forall\varepsilon > 0\, \forall n \geq \Phi(\varepsilon, N, b, K)\ (\|T^{n+1}x - T^n x\| < \varepsilon).$$

Theorem (K. FoCM 2019)

$$\Phi(\varepsilon, N, b, K) := \left\lceil \frac{18b + 12\alpha(\varepsilon/6))}{\varepsilon} - 1 \right\rceil \left\lceil \left(\frac{D}{\omega(D, \tilde{\varepsilon})} \right) \right\rceil$$

is a **rate of asymptotic regularity** in Bauschke's result, where

$$\tilde{\varepsilon} := \frac{\varepsilon^2}{27b + 18\alpha(\varepsilon/6)}, \quad D := 2b + NK, \quad \omega(D, \tilde{\varepsilon}) := \frac{1}{16D}(\tilde{\varepsilon}/N)^2.$$

$$\alpha(\varepsilon) := \frac{(K^2 + N^3(N-1)^2K^2)N^2}{\varepsilon}.$$

Here $b \geq \|x\|$ and $K \geq \left(\sum_{i=1}^{N} \|c_i\|^2\right)^{\frac{1}{2}}$ for some $(c_1, \ldots, c_N) \in C_1 \times \ldots \times C_N$.

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A. Sipoș (2020): Extension to general averaged mappings.

Proximal mappings in Banach spaces

In Bačák/K. J. Convex Analysis 2018, we give a quantitative treatment of the **proximal mapping in uniformly convex Banach spaces** $X, f : X \to (-\infty, \infty]$ proper lsc convex, $\lambda > 0, \Phi$ a Young function:

$$\mathrm{prox}^{\Phi}_{\lambda,f}(x) := \operatorname*{argmin}_{y \in X} \left[f(y) + \frac{1}{\phi(\lambda)} \Phi\left(\|x - y\| \right) \right], \quad x \in X.$$

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Classically: X Hilbert space, $\Phi(t) := \frac{1}{2}t^2, \phi(t) = t$. Then

$$\mathrm{prox}_{\lambda,f}(x) := \underset{y\in X}{\mathrm{argmin}} \left[f(y) + \frac{1}{2\lambda}\|x-y\|^2\right] = (I+\lambda\partial f)^{-1}(x), \quad x \in X,$$

i.e. $\mathrm{prox}_{\lambda,f}$ is the resolvent of the maximally monotone subdifferential ∂f and hence the **metric projection** P_C for closed convex C if f is the **indicator function** of C.

Nonexpansive if X Hilbert space (this characterizes Hilbert spaces).

For the above generalization to arbitrary Young functions and in uniformly convex spaces.

Theorem (Bačák/K., J. Convex Anal. 2018)

For all $z \in X, r > 0, \varepsilon > 0, x, y \in B(z,r)$ and $\lambda \in (0,1]$:

$$\|x-y\| < \delta(\varepsilon) \implies \left\|\mathbf{prox}^{\Phi}_{\lambda,f}(x) - \mathbf{prox}^{\Phi}_{\lambda,f}(y)\right\| < \varepsilon, \text{ where}$$

$$\delta(\varepsilon) := \min\left\{\frac{\varepsilon}{2}, \frac{2}{\phi(R)}\delta_R\left(\frac{\varepsilon}{2}\right)\right\},$$

and $R > 0$ is a constant independent of $\lambda \in (0,1]$ and δ_R is a modulus of uniform convexity of $\Phi \circ \|\cdot\|$ on the ball $B(0,R)$.

Important **qualitative consequence**: modulus does not depend on λ. This is not true for an alternative definition of proximal mappings due to Penot and Ratsimahalo and supports our definition!

The proximal point algorithm

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In general: **strong convergence** (even in infinite dimensional Hilbert spaces) **only for** so-called **Halpern type variant of PPA**:

$$x_{n+1} := \alpha_n u + (1-\alpha_n) J_{\lambda_n A} \quad \textbf{(HPPA)}$$

(necessary conditions: $\lim \alpha_n = 0, \sum \alpha_n = \infty$).

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The proofs and their resp. minings are very different!

Viscosity approximation method

Consider for bounded convex $C \subset X$, X normed or hyperbolic space, $T : C \to C$ nonexpansive and suitable $(\alpha_n) \subset (0,1), u \in C$ the Halpern iteration

$$w_0 \in C, \quad w_{n+1} = (1-\alpha_n)T(w_n) + \alpha_n u, \tag{1}$$

and its viscosity generalization for contractions ϕ

$$x_0 \in C, \; x_{n+1} = (1-\alpha_n)T(x_n) + \alpha_n \phi(x_n). \tag{2}$$

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K./Pinto 2020 showed that a uniform rate of metastability θ for $(w_n(u)$ translates into $\Psi[\theta]$ with

$$\forall N \in \mathbb{N}\, \forall \varepsilon > 0\, \exists n \in [N, \Psi[\theta](\varepsilon, g, N)]\, \exists k \in [N, n], z \in C$$
$$\left(d(z, w_k(\Phi(z))) \leq \varepsilon \wedge \forall i \in [n, n+g(n)]\, (d(x_i, z) \leq \tfrac{\varepsilon}{2})\right).$$

Fejér monotonicity and regularity

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Definition

A sequence (x_n) in a metric space (X, d) is Fejér monotone w.r.t. a subset $S \subseteq X$ if $\forall n \in \mathbb{N}\, \forall p \in S\ (d(x_{n+1}, p) \leq d(x_n, p))$.

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Why is this important?

If one has metric regularity one not only gets strong convergence but even a **rate of convergence!**

Moduli of regularity for mappings

In continuous optimization notions of **linear** or **Hölder metric regularity**, **error bounds** and **weak sharp minima** etc. play an important role which can be viewed as (often local forms of) special cases of (see also R.M. Anderson: 'Almost' implies 'Near', TAMS 1986):

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Definition (K./Lopéz-Acedo/Nicolae, Israel J. Math 2019)

Let $zer\, F := \{x \in X : F(x) = 0\} \neq \emptyset$. F is **regular** w.r.t. $zer\, F$ if

$$\forall n \in \mathbb{N} \exists k \in \mathbb{N}\, \forall x \in X(|F(x)| < 2^{-k} \rightarrow \exists z' \in zer\, F(d(x, z') < 2^{-n})).$$

A function $\omega : \mathbb{N} \rightarrow \mathbb{N}$ providing $k = \omega(n)$ is a **modulus of regularity**.

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This covers fixed point and equilibrium problems as well as zeros of set-valued operators.

Computational use of moduli of regularity

Proposition (K./Lopéz-Acedo/Nicolae Israel J. Math. 2019)

Let $F : X \to \mathbb{R}$ *be with* $zer\ F \neq \emptyset$ *and with modulus of metric regularity* ω. *Let* (x_n) *be a sequence in* X *and* $\psi : \mathbb{N} \to \mathbb{N}$ *be s.t.*

$$\forall k \in \mathbb{N}\, \exists n \leq \psi(k)\ \left(|F(x_n)| < 2^{-k}\right),$$

where (x_n) *is Fejér monotone w.r.t.* $zer\ F$. *Then* (x_n) *is Cauchy:*

$$\forall k \in \mathbb{N}\, \forall n, \tilde{n} \geq \Phi(k) := \psi(\omega(k+1))\ \left(d(x_n, x_{\tilde{n}}) < 2^{-k}\right)$$

and $$\forall k \in \mathbb{N}\, \forall n \geq \Phi(k)\ \left(dist(x_n, zer\ F) < 2^{-k}\right).$$

If X *is complete and* F *is continuous, then* $\lim x_n \in zer\ F$.

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In general, there will be no computable moduli of metric regularity:

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There exists a **computable firmly nonexpansive** mapping $T : [0,1] \to [0,1]$ which has **no computable modulus** of metric regularity ϕ w.r.t. $Fix(T)\,(= zer\,(I - T))$.

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In fact, the cases where one can compute such a modulus are rare. However there are important cases where this is true (connection to o-minimality: tame optimization, Ioffe, Lewis, Bolte, Daniilidis...!)

Applications to Pursuit-evasion games

Let (X,d) be a uniquely geodesic space, $D>0$. $L_0, M_0 \in A$ starting points of the lion L and the man M. After n-steps, M moves to any point M_n s.t. $d(M_n, M_{n+1}) \leq D$ and L moves via the geodesic $[L_n, M_n]$ s.t. $d(L_n, L_{n+1}){=}\min\{D, d(L_n, M_n)\}$.

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'$\boldsymbol{\lim d(L_{n+1}, M_n) = 0}$' $\in \forall\exists$ since the sequence is nonincreasing!

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Details in K./López-Acedo/Nicolae in Pacific J. Math. 2021.

Betweenness and uniform betweenness in metric spaces

Definition (Diminnie and White 1981)

Let (X, d) be a metric space. X satisfies the betweenness property if for any distinct points $x, y, z, w \in X$

$$\left.\begin{array}{l} d(x,y) + d(y,z) \leq d(x,z) \\ d(y,z) + d(z,w) \leq d(y,w) \end{array}\right\} \Rightarrow d(x,z) + d(z,w) \leq d(x,w).$$

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For normed spaces, betweenness follows from (but is strictly weaker than) strict convexity. It fails for $(\mathbb{R}^2, \|\cdot\|_\infty), (\mathbb{R}^2, \|\cdot\|_1)$ but holds for some nonstrictly convex spaces.

The functional interpretation upgrades betweenness to (equivalent in the compact case!):

Definition (K., Lopéz-Acedo, Nicolae, to appear in: Pacific J. Math.)

A metric space (X, d) satisfies the uniform betweenness property with modulus $\Theta : (0, \infty)^3 \to (0, \infty)$ if

$$\forall \varepsilon, a, b > 0\, \forall x, y, z, w \in X \left(\left\{ \begin{array}{l} \mathrm{sep}\{x, y, z, w\} \geq a \wedge \mathrm{diam}\{x, y, z, w\} \leq b \\ d(x, y) + d(y, z) \leq d(x, z) + \Theta(\varepsilon, a, b) \\ d(y, z) + d(z, w) \leq d(y, w) + \Theta(\varepsilon, a, b) \end{array} \right\} \begin{array}{c} \\ \\ \\ \Rightarrow d(x, z) + d(z, w) \leq d(x, w) + \varepsilon \end{array} \right).$$

Theorem (K./Lopéz-Acedo/Nicolae, to appear in: Pacific J. Math.)

Let X be a bounded metric space with the uniform betweenness property and $\langle (M_n), (L_n) \rangle$ be a Lion-Man game, speed $D > 0$.
Then the Lion approaches the man arbitrarily close.

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Moreover with

$b \geq \mathrm{diam}(X)$, $\Theta(\varepsilon) := \Theta(\varepsilon,\varepsilon,b)$, $N\in\mathbb{N}$, $b+1<ND$:

$$\forall \varepsilon>0\,\forall n\geq \Omega_{D,b,\Theta}(\varepsilon)\ (d(L_{n+1},M_n)<\varepsilon),$$

where

$$\Omega_{D,b,\Theta}(\varepsilon) = N + N\left\lceil \frac{b}{\Theta^{(N)}(\alpha)}\right\rceil$$

with

$$0<\alpha\leq \min\left\{\frac{1}{N},\frac{D}{2},\frac{\varepsilon}{2}\right\}.$$

Moduli of uniform betweennes

Θ can be **explicitly computed** for L^p $(1 < p < \infty)$ (of order 2 if $1 < p < 2$ and of order p if $2 \leq p < \infty$) and **CAT(κ)-spaces, $\kappa > 0$** (of order 2).

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Low complexity Θ's can also be obtained in a number of **non-uniquely geodesic** normed and metric cases!

Applications to Ergodic Theory

The Mean Ergodic Theorem

X **Hilbert space**, $f : X \to X$ **linear** and $\|f(z)\| \leq \|z\|$ for all $z \in X$.

$$\mathbf{A_n(z) := \frac{1}{n+1} S_n(z),\ \text{where } S_n(z) := \sum_{i=0}^{n} f^{(i)}(z) \quad (n \geq 0)}$$

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Theorem (von Neumann Mean Ergodic Theorem)

For every $z \in X$, the sequence $(A_n(z))_n$ converges.

The Mean Ergodic Theorem

X **Hilbert space**, $f: X \to X$ **linear** and $\|f(z)\| \leq \|z\|$ for all $z \in X$.

$$A_n(z) := \frac{1}{n+1} S_n(z), \text{ where } S_n(z) := \sum_{i=0}^{n} f^{(i)}(z) \quad (n \geq 0)$$

Theorem (von Neumann Mean Ergodic Theorem)

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

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$f^* := I$ majorizes f: $f(0) = 0$.

Theorem (K./Leuştean, Ergodic Theor. Dynam. Syst. 2009)

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where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^{(K)}(0), \text{ with}$$
$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$
$$h, \tilde{h} : \mathbb{N} \to \mathbb{N},\ h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \leq n} h(i).$$

Recently generalized by Sipoş to multi-parameter version.

Bounding the number of fluctuations

We say that (x_n) admits k ε-fluctuations if there are $i_1 \leq j_1 \leq \ldots i_k \leq j_k$ s.t. $\|x_{j_n} - x_{i_n}\| \geq \varepsilon$ for $n = 1, \ldots, k$.

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$(A_n(z))$ admits at most

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For the Hilbert space case: first fluctuation bounds by Jones, Ostrovskii, Rosenblatt 1996.

Other applications of proof mining in analysis

- Abstract Cauchy problems: Koutsoukou-Argyrakis/K. JMAA 2015.
- Best approximation theory: Chebycheff (K. NFAO 1993), L^1 (K./Oliva APAL 2003), Chebycheff for polynomials with restricted coefficients (Sipoș Math. Nach. 2021).
- Tauberian theorems (Powell MLQ 2020).
- Financial mathematics (Oliva 2020)

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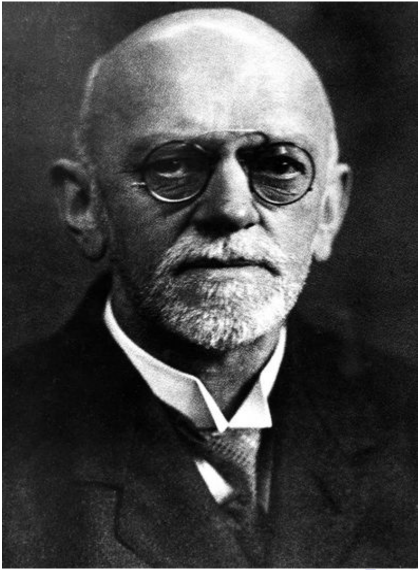
David Hilbert (1862-1943)



Surveys

- Kohlenbach, U., Recent progress in proof mining in nonlinear analysis. IFCoLog Journal of Logics and their Applications **10**, Issue 4, pp. 3357-3406 (2017).
- Kohlenbach, U., Proof-theoretic Methods in Nonlinear Analysis. In: Proc. ICM 2018, B. Sirakov, P. Ney de Souza, M. Viana (eds.), Vol. 2, pp. 61-82. World Scientific 2019.