Ramsey's Theorem, Reverse Mathematics and Their foundational Issues

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Computation Models and Recursive Sets

- Computability can be defined mathematically, by Turing machines for example.
- $A \subseteq \mathbb{N}$ is recursive or computable if we can decide its membership by a (Turing) program.
- That is, for any input n to the program, it must give us a Y/N answer according to if $n \in A$.
- For example, the set of prime numbers is recursive; so is $\{2^n : n \in \mathbb{N}\}$.
- **Q**: Are there any non-recursive sets?

From Computable to Non-Computable

- All Turing machines (or programs) can be listed (enumerated) in an effective way: denoted by $\Phi_0, \Phi_1, \ldots$.
- (To achieve this, we have to allow "bad" programs, such as, wrong grammar, or looping forever.)
- But, all good programs are included. Thus
- If a set *A* is recursive/computable, it must be computed by one of these Φ_e's.

Halting Problem is Undecidable

Theorem (Turing, 1936)
The set $\emptyset' = \{e : \Phi_e(e) \text{ is defined}\}$ is not recursive.

Proof. Suppose that there is a program Ψ computes it, then the following function is computable:

$$d(e) = \begin{cases} \Phi_e(e) + 1, & \text{if } \Psi \text{ says that } e \in \emptyset'; \\ 2021, & \text{otherwise.} \end{cases}$$

But this function d disagrees with function computed by Φ_e for

every e (because $d(e) \neq \Phi_e(e)$ for all e). Contradiction. Q.E.D.

Remarks after Halting Problem

- It is an effective version of Cantor's diagonal argument.
- (It is the partialness that allows us to define computability in its most general form.)
- $\emptyset'$ can be defined by an existential numerical quantifier:

$$e \in \emptyset' \iff \exists s[\Phi_e(e) \text{ halts at stage } s].$$

- (So it is not that far away from computable. It is so-called *recursively enumerable*.)

Relative Computation

- (Occasionally I will mention relative computation.)
- For example, although $\emptyset'$ is not computable, but *if* we can retrieve all information of $\emptyset'$, we can compute more sets.
- But we also have relativized halting problem $\emptyset'' = (\emptyset')'$, which is not computable even if we have the complete information of $\emptyset'$.
- Thus we have more and more complicated sets, which brings us the arithmetical hierarchy.

Arithmetical Hierarchy

- Language of first order Peano Arithmetic: 0, S, $+$, $\times$, $<$.
- Formulas are classified by the number of alternating blocks of quantifiers: Σ^0_n and Π^0_n.
- (Δ^0_n: having two equivalent forms, one Σ^0_n, one Π^0_n.)
- For example, to say finite (or infinite) requires two quantifiers.
- Each formula $\varphi(v)$ defines a subset of $\mathbb{N}$. Definable sets are classified by their defining formulas.
- First-order definable sets are called arithmetical sets.

From First Order to Second Order

- First order: variables and quantifiers are ranging over individuals (i.e. numbers).

- For example, $\forall m \exists n (n > m)$. It is in fact Π^0_2.

  $\emptyset'$ is Σ^0_1.

  Fermat Last Theorem $(\forall x, y, z \neq 0 \forall n \geq 3)[x^n + y^n \neq z^n]$ is Π^0_1.

- Second order: variables and quantifiers can range over sets or relations. For example, "every nonempty subset has a least element", which is of the form $\forall X(X \neq \emptyset \rightarrow ...)$, so it is Π^1_1.

RCA_0 and ACA_0

- Reverse mathematics uses fragments of Second Order Arithmetic.
- RCA_0: Σ^0_1-induction and Δ^0_1-comprehension:
  For $\varphi \in \Delta^0_1$, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.
- (WKL_0: RCA_0 and every infinite binary tree has an infinite path.)
- ACA_0: RCA_0 and for φ arithmetical, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.
- (together with ATR_0 and Π^1_1-CA_0, they form the "*big five*".)

Models in Reverse Math

- A *model* $\mathcal{M}$ *of second-order arithmetic* consists $(M, 0, S, +, \times, <, \mathcal{X})$ where $(M, 0, S, +, \times, <)$ is its first-order part and the set variables are interpreted as members of $\mathcal{X}$.
- Models of RCA_0: Its second-order part $\mathcal{X}$ is closed under $\leq_T$ and Turing join, namely a *Turing ideal*.
- In the (minimal) model of RCA_0, $M = \mathbb{N}$ and $\mathcal{X}$ only consists of recursive subsets.
- In the (minimal) model of ACA_0, $M = \mathbb{N}$ and $\mathcal{X}$ only consists arithmetical subsets.

(Why "Reverse"?)

- Reverse Mathematics compares the strengths of math theorems, can be viewed as a classification project.
- Typical scenario: Given theorems P and Q, decide $P \Rightarrow Q$ or $P \not\Rightarrow Q$.
- The way to show that $P \not\Rightarrow Q$ is to "make" P true and Q false. But all mathematical theorems are all true.
- Thus we have to work in some weaker systems Γ, and demonstrate that "Γ proves P but not Q".
- To be more systematic, instead of comparing them pairwise, we will have a hierarchy of systems $\Gamma_0 < \Gamma_1 < \dots$, and Γ_i proves P but not Q (or better, Q proves Γ_j for some $j > i$).
- (notice the reverse direction.)

Remarks on Axioms in Reverse Math

- They all assert the existence of certain sets.
- Some are measured by syntactical complexity, e.g. RCA_0 or ACA_0.
- (Some are from the analysis of mathematical tools, e.g. WKL_0 corresponds to Compactness Theorem.)
- Weaker system "allows" less sets. For example, RCA_0 is strictly weakly than ACA_0.

Ramsey's Theorem

For $A \subseteq \mathbb{N}$, let $[A]^n$ denote the set of all n-element subsets of A.

Theorem (Ramsey 1930) Any $f : [\mathbb{N}]^n \to \{0, 1, \ldots, k-1\}$ has an infinite *homogeneous set* $H \subseteq \mathbb{N}$, namely, f is constant on $[H]^n$.

Many of combinatorial theorems are of Π^1_2 form $(\forall X \exists Y(\ldots))$.
Convention: Every coloring problem has a homogeneous solution.

Notation: The version above is denoted by RT^n_k.

Examples

- RT^1_k is also called Pigeonhole Principle.
- An example of RT^2_2: Define $f : [\mathbb{N}]^2 \to \{0, 1\}$ as follows. Suppose that $x < y$, then

  $$f(x, y) = \begin{cases} 0, & \text{if } x, y \text{ are coprime} \\ 1, & \text{otherwise} \end{cases}$$

  Then the set of all prime numbers is a homogeneous set with color 0 and $\{2, 4, 8, 16, \dots\}$ is another homogeneous set with color 1.
- Observation: For $k > 2$, RT^n_k can be reduced to RT^n_2.
- Thus we only need to show RT^n_2.

A Proof of RT^2_2

- We follow Ramsey's original idea. (Ramsey used induction on n to show RT^n_2.)

  For simplicity, we only go from RT^1_2 to RT^2_2 (instead of going from k to $k+1$).

- Statement: If we color pairs of natural numbers in two colors (Red and Blue), then there is an infinite subset $H \subset \mathbb{N}$, such that any pair formed by elements in H is colored by the same colour.

- **Idea**: Keep "purifying" the tail w.r.t. the head. Tool: RT^1_2.

Proof (conti.)

- Step 0: Let $a_0 = 0$. (a_0 is the current “head”).
- For each $n > 0$, the pair $(0, n)$ is either colored red or blue. By RT^1_2, either red or blue must get infinitely many elements; say red.
- Discard all “blue” numbers. The tail is “purified” so that every number when paired with a_0 has color red. Denote the tail by T_0.
- In picture:

$$\bullet(a_0) \quad \underbrace{b_0, b_1, b_2, \ldots}_{T_0}$$

Proof (conti.)

- Step 1: Let $a_1 = b_0$. (a_1 is now the "head").
- For each $n \in T_0$, the pair (a_1, n) is either colored red or blue. By RT^1_2, either red or blue must get infinitely many elements; say blue.
- Discard all "red" numbers from T_0. The tail is "purified" so that every number when paired with a_1 has color blue. Denote the tail by T_1.
- In picture:

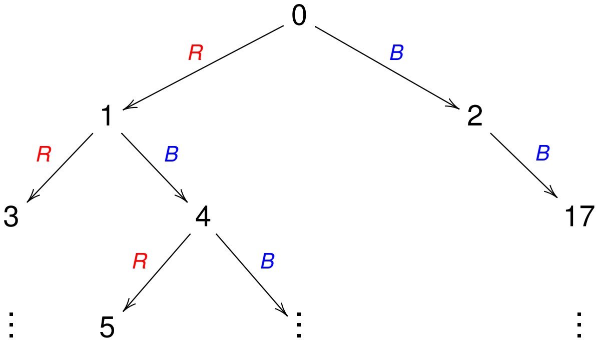
$$\bullet(a_0), \bullet(a_1) \quad \underbrace{c_0, c_1, c_2, \ldots}_{T_1}$$

Proof (conti.)

- Keep doing this, (we will pass all finite stages).
- In the end, we get an infinite sequence (a_n) and each a_i is associated with either red or blue depending how it is colored with elements in the tail T_i.
- Use RT^1_2 one more time, we get one infinite subset H_1 which is associated with one color.
- H_1 is what we wanted.

Another Proof of RT^2_2

Proof (idea). We enumerate a tree based on the coloring as illustrated by the following example:



(The rules of the enumeration are described verbally.)

Sketch of Proof (conti.)

- We obtain a finite-branching infinite tree.
- So there is an infinite branch of the tree. (This fact is called König Lemma.) For example, the path $0 \to 1 \to 4 \to 5 \to \cdots$.
- We can then read from the branch a homogeneous set depending on whether we see infinitely many Red label or Blue label.
- (Detail: If the path has infinitely left turns, then let H_2 be the set of nodes where it turns left.)

Analysis of the Two Proofs

- **Q**: Are the two proofs the same?
- In proof one: We first get a sequence of sets $T_0 \supset T_1 \supset \dots$.
- T_0 requires the information which color is infinite (when paired with 0); so it is of “two quantifiers” (above the complexity of the coloring function).
- T_1 is another two quantifiers above T_0, so it is roughly four quantifiers.
- Thus even the sequence (T_n) may not be definable in arithmetic; not to mention the final use of RT^1_2 to (a_n).

Analysis of the Two Proofs (conti.)

- Now for the second proof, the binary tree is enumerated, so it is Σ^0_1 definable (over the coloring function).
- It takes at most two more quantifiers to pick out an infinite branch.
- And at most another two more quantifiers to pick out H_2.
- The bottom line is H_2 is definable in arithmetic.
- **Theorem (Jockusch 1972)** $\mathrm{ACA}_0 \vdash \mathrm{RT}^n_k$.
- Informal reading: If a model contains all arithmetical sets, then it must contain all solutions of (arithmetical instances of) RT^n_k.

The Reversal of RT^3_2

Theorem (Jockusch 1972) $\mathrm{RT}^3_2 \vdash \mathrm{ACA}_0$.

- Informal reading: If a model contains all solutions of RT^3_2, then it is closed under Turing jump.
- We will define a recursive 2-coloring g of triples, whose solutions always computes the halting problem.
- With relativization, it will show Jockusch's Theorem.

Coding in Halting Problem

Define $g : [\mathbb{N}]^3 \to \{0, 1\}$ by: For $a < b < c$, set

$$g(a, b, c) = \begin{cases} 0, & \text{if } \emptyset' \upharpoonright a \text{ changes} \\ & \text{between stages } b \text{ and } c; \\ 1, & \text{otherwise.} \end{cases}$$

g is recursive.

Now suppose H is an infinite homogenous set for g.

Then H can't be 0-homogenous (because for a fixed a, $\emptyset' \upharpoonright a$ will be "stable" eventually).

Proof (conti.)

- Let's use H to compute $\emptyset'$ as follows:
- Given any e, first find an $a \in H$ with $a > e$. Then pick $b \in H$ with $a < b$.
- Then $e \in \emptyset'$ iff $e \in \emptyset'$ by the stage b. (Because for any $c \in H$ with $b < c$, $g(a, b, c) = 1$ which means $\emptyset' \upharpoonright a$ has been "stable" at stage b.)

What about RT^2_2?

- **Q**: Can we do the similar coding using pairs instead of triples?
- **Theorem (Seetapun and Slaman, 1995)** RT^2_2 does not imply ACA_0.
- Also **Theorem (Jockusch, 1972)** WKL_0 does not imply RT^2_2.
- (Recall that WKL_0 is one of the "big five" which is strictly between RCA_0 and ACA_0.)
- **Theorem (Liu, 2012)** RT^2_2 does not imply WKL_0.

A Brief Summary

- Both recursion theory and reverse mathematics share the “hierarchical’ view of math.
- They tried to set up systems to measure the complexity (the gap between the “problem” and the “solution”).
- (also to measure abstract concepts.)
- Wish: Turning more parts of human knowledge into exact science.

First Non-math Remark/Question

- Informal reading of Ramsey's Theorem: "Absolute chaos is impossible".
- Every (sufficiently large) irregular structure contains a (large) highly regular substructure.
- Thus Ramsey's Theorem allows us to see "order" (the homogenous set) through "chaos" (coloring).
- **Q1**: Is this order "real"? Is it "objective"?

Realism vs. Others

- Of course, Q1 depends on the meaning of “real” or “existence”.
- Platonism/Realism would answer Q1 positively.
- Let’s consider finitism.
- (Fictionism sounds a bit paradoxical to me: If there is no meaning/truth for math statements, what’s the point of studying Philosophy of Math?)

The word “finitary”

- It comes from Hilbert and his Program.
- “Hilbert’s program called for using only the most secure methods, which in German he called “finit”, usually translated as “finitary” (after Kleene 1952) and occasionally as “finitistic”. They can be characterized as methods not using any completed infinity; i.e., no objects themselves infinite are to be used, and only potentially infinite collections of them, like the natural number sequence 0,1,2,... considered as unbounded above but not as a completed collection.” (Kleene)

Finitism

- Tait (1981): finitism =PRA
- RT^n_k is beyond PRA.
- We could study finite versions of Ramsey's Theorem (Some of them went beyond PRA, e.g., Paris-Harrington Theorem).
- Or take formalist approach: Identify coloring and solution with their defining formulas (they are finite strings) and turn this problem into a finite combinatorial problem.
- But these will not satisfy either camps.

Digression: Finitary Mathematics

- Wiki: "it is widely agreed that all reasoning of PRA is finitist. Many also believe that all of finitism is captured by PRA."
- Simpson 1988, *Partial Realizations of Hilbert's Program*. (Recommended.)
- Ultrafinitism? "Only in our own time has there arisen an ultrafinitist school which posits bounds on the length of the natural number sequence. And the ultrafinitists have neither refuted finitistic mathematics nor shown us what an ultrafinitist textbook would look like." (Simpson 1988)
- On the other hand, some intuitionists think that finitary "$\sim$" $< \varepsilon_0$ whereas PRA "$\sim$" ω^ω.

Quote from Gödel 1972

- At any rate Bernays' observations...teach us to distinguish two component parts in the concept of finitary mathematics, namely: first, the constructivistic element, which consists in admitting reference to mathematical objects or facts only in the sense that they can be exhibited, or obtained by construction or proof; second, the specifically finitistic element, which requires in addition that the objects and facts considered should be given in concrete mathematical intuition. This, as far as the objects are concerned, means that they must be finite space-time configurations of elements whose nature is irrelevant except for equality or difference.

- "It is the second requirement which must be dropped."

A Realism

- Platonists also agree that math objects do not exist in space-time.
- “they would have to be assumed at least in the same sense as any well-established physical theory.” (Gödel 1947)
- (By changing from $\exists$ to $\forall$, we see the other side of the problem.)
- Example: The expansion of set theoretic universe. Gödel’s example: Inaccessible cardinal “means nothing else but that the totality of sets obtainable by use of the procedures of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for further applications of these procedures).” (Gödel 1964)

Second Non-math Remark/Observation

- The solution is more complicated than the problem.
- Reverse Math provides us lots of examples, with rigorous proofs.
- We have to get more and more abstract.
- Gödel thought that incompleteness is to be overcome by the development of human understanding through the use of “more and more abstract terms”.
- (This is the reason that we reject Finitism.)

Digression: The debate between Gödel and Turing

What Turing disregards completely is the fact that mind, in its use, is not static, but constantly developing, i.e., that we understand abstract terms more and more precisely as we go on using them, and that more and more abstract terms enter the sphere of our understanding. There may exist systematic methods of actualizing this development, which could form part of the procedure. Therefore, although at each stage the number and precision of the abstract terms at our disposal may be finite, both (and, therefore, also Turing's number of distinguishable states of mind) may converge toward infinity in the course of the application of the procedure.

Third Non-Math Remark/Observation

- Platonism: The rules/orders are real and we can raise our levels to see them.
- Can we go further?
- Hilbert: Wir müssen wissen, wir werden wissen!
- Rationalistic Optimism: it is not the case 'that human reason is utterly irrational by asking questions it cannot answer, while asserting emphatically that only reason can answer them'. (Wang Hao quoting Gödel)
- (Not only there are laws of the universe, physical or mathematical, they are "simple".)
- **Q**: Does Rationalistic Optimism have verifiable consequences?

Gödel 1951

So the following disjunctive conclusion is inevitable:

Either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine,

or else there exist absolutely unsolvable diophantine problems of the type specified

(where the case that both terms of the disjunction are true is not excluded, so that there are, strictly speaking, three alternatives).

It is this mathematically established fact which seems to me of great philosophical interest.

Conclusion/Homework

- (For Platonists) Demonstrate a unified foundation for all sciences/human knowledge.
- (For Rationalistic Optimists) Demonstrate that human mind far surpasses the finite machines.