Finding the limit of incompleteness

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Modern logic and Philosophy

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Research Motivation Understanding incompleteness:
Exploring the relationship between incompleteness, self-reference, provability logic, logical paradox and formal theory of truth.

Part One: the current state of research on incompleteness

Outline of the current state of research

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(1) Clarifications of misinterpretations of Gödel's incompleteness theorem

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(6) The intensionality of **G2** for **PA**

(7) Incompleteness and provability logic

(8) Incompleteness for higher order arithmetic

Gödel's incompleteness theorem

Two goals of Hilbert's program:

Completeness A proof that all true mathematical statements can be proved in the formalism of mathematics.

Consistency A proof that no contradiction can be obtained in the formalism of mathematics using only "finitistic" reasoning about finite mathematical objects.

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Theorem 1 (Gödel-Rosser)

(1) *Gödel-Rosser first incompleteness theorem* (G1)*: If T is a recursively axiomatized consistent extension of* **PA***, then T is not complete (there is a sentence θ such that $T \nvdash \theta$ and $T \nvdash \neg\theta$).*

(2) *Gödel's second incompleteness theorem* (G2)*: If T is a recursively axiomatized consistent extension of* **PA***, then the consistency of T is not provable in T.*

Arithmetization

- The three main ideas in Gödel's proof of G1 and G2 are arithmetization of the syntax of **PA**, representability of recursive functions in **PA** and self-reference construction.
- Under arithmetization, any formula or finite sequence of formulas of the theory can be coded by a natural number (called Gödel's number). Under Gödel's arithmetization, the set of Gödel's number of axioms of **PA** is recursive. We use $\ulcorner \phi \urcorner$ to denote the numeral in $L(\mathbf{PA})$ of the Gödel number of ϕ.
- Then we could define some relations on natural numbers which express metamathematical property of **PA**. Define $\mathbf{Prf_{PA}}(m, n)$ iff n is the Gödel's number of a proof of the formula with Gödel number m in **PA**. We can show that $\mathbf{Prf_{PA}}(m, n)$ is recursive.

Representability

- A n-ary relation $R(x_1, \cdots, x_n)$ on $\mathbb{N}^n$ is representable in T iff there is a formula $\phi(x_1, \cdots, x_n)$ such that $T \vdash \phi(\overline{m_1}, \cdots, \overline{m_n})$ if $R(m_1, \cdots, m_n)$ holds; and $T \vdash \neg\phi(\overline{m_1}, \cdots, \overline{m_n})$ if $R(m_1, \cdots, m_n)$ does not hold.
- Gödel proves that every recursive relation is representable in **PA** and hence there is a formula $\phi(x, y)$ which represents $\mathbf{Prf_{PA}}(m, n)$ in **PA**.
- From the representation formula $\phi(x, y)$, we could define the provability predicate $\mathbf{Pr}(x)$ as $\mathbf{Pr}(x) \triangleq \exists y\, \phi(x, y)$.
- $\mathbf{Pr}(x)$ satisfies the following conditions:
  (1) If $\mathbf{PA} \vdash \varphi$, then $\mathbf{PA} \vdash \mathbf{Pr}(\ulcorner\varphi\urcorner)$;
  (2) $\mathbf{PA} \vdash \mathbf{Pr}(\ulcorner\varphi\urcorner) \rightarrow (\mathbf{Pr}(\ulcorner\varphi \rightarrow \psi\urcorner) \rightarrow \mathbf{Pr}(\ulcorner\psi\urcorner))$;
  (3) $\mathbf{PA} \vdash \mathbf{Pr}(\ulcorner\varphi\urcorner) \rightarrow \mathbf{Pr}(\ulcorner\mathbf{Pr}(\ulcorner\varphi\urcorner)\urcorner)$.

Self-reference construction

- Gödel constructs a Gödel sentence $\mathbf{G}$ which asserts its own unprovability in $\mathbf{PA}$, i.e. $\mathbf{PA} \vdash \mathbf{G} \leftrightarrow \neg\mathbf{Pr}(\ulcorner \mathbf{G} \urcorner)$.
- Gödel shows that if $\mathbf{PA}$ is consistent, then $\mathbf{G}$ is not provable in $\mathbf{PA}$; and if $\mathbf{PA}$ is ω-consistent, then $\neg\mathbf{G}$ is not provable in $\mathbf{PA}$.
- Define $\mathbf{Con}(\mathbf{PA}) \triangleq \neg\mathbf{Pr}_{\mathbf{PA}}(\ulcorner \mathbf{0} = \mathbf{1} \urcorner)$.
- From derivability conditions, we can show that $\mathbf{PA} \vdash \mathbf{Con}(\mathbf{PA}) \leftrightarrow \mathbf{G}$.
- So G2 holds: if $\mathbf{PA}$ is consistent, then $\mathbf{PA} \nvdash \mathbf{Con}(\mathbf{PA})$.

The notion of interpretation

- An interpretation of a theory T in a theory S is a mapping from formulas of T to formulas of S that maps all axioms of T to sentences provable in S.
- Let $S \trianglelefteq T$ denote that S is interpretable in T (or T interprets S); $S \lhd T$ denotes that S is interpretable in T but T is not interpretable in S; S and T are mutually interpretable if $S \trianglelefteq T$ and $T \trianglelefteq S$.
- G1 can be generalized via interpretability: there exists a weak recursively axiomatizable consistent sub-theory T of **PA** such that for any recursively axiomatizable theory S, if T is interpretable in S, then S is incomplete.

Misinterpretation Any theory of arithmetic is incomplete.

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- Whether a theory of arithmetic is complete depends on the language of the theory.
- **PA** is incomplete in the language of $L(\mathbf{0}, \mathbf{S}, +, \cdot)$. But there are respectively recursively axiomatized complete arithmetic theories in the language of $L(\mathbf{0}, \mathbf{S})$, $L(\mathbf{0}, \mathbf{S}, <)$ and $L(\mathbf{0}, \mathbf{S}, <, +)$.
- Containing the arithmetic of multiplication is essential for the proof of G1. For example, Presburger arithmetic is the theory of arithmetic of addition in $L(\mathbf{0}, \mathbf{S}, +)$; but Presburger arithmetic is complete.
- Containing the arithmetic of multiplication is not sufficient for being incomplete. For example, there is a complete recursively axiomatized theory in the language $L(\mathbf{0}, \mathbf{S}, \cdot)$.

Misinterpretation Since **PA** is not complete and $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, theories about integers, rational numbers and real numbers are all incomplete.

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- Since $Th(\mathbb{N}, +, \cdot)$ is interpretable in $Th(\mathbb{Z}, +, \cdot)$, $Th(\mathbb{Z}, +, \cdot)$ is undecidable and hence not recursive axiomatizable but it has a finitely axiomatizable incomplete sub-theory of integers.
- Since $Th(\mathbb{N}, +, \cdot)$ is interpretable in $Th(\mathbb{Q}, +, \cdot)$, $Th(\mathbb{Q}, +, \cdot)$ is undecidable and hence not recursive axiomatizable but it has a finitely axiomatizable incomplete sub-theory of rational numbers.
- But $Th(\mathbb{R}, +, \cdot)$ is decidable, recursively axiomatizable theory (even if not finitely axiomatizable) and $Th(\mathbb{R}, +, \cdot) =$ **RCF** (the theory of real closed field).
- This fact does not contradict G1 since none of $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ is definable in the structure $(\mathbb{R}, +, \cdot)$.

Misinterpretation Any consistent extension of **PA** is incomplete.

Definition 1

Let T be a theory and Γ be a class of formulas.

(1) *T is Σ_n-definable iff there is a Σ_n formula $\alpha(x)$ such that n is the Gödel number of some sentence of T if and only if $\mathfrak{N} \models \alpha(\overline{n})$ where $\mathfrak{N} = (\mathbb{N}, +, \cdot)$.*

(2) *T is Σ_n-sound if and only if for all Σ_n sentences ϕ, if $T \vdash \phi$, then $\mathfrak{N} \models \phi$.*

(3) *T is Γ-decisive if and only if for all Γ sentences ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$ holds.*

From G1, if T is a consistent Σ_1-definable extension of **PA**, then T is not Π_1-decisive.

The generalization of G1 to arithmetically definable theory

Kikuchi and Kurahashi generalized G1 to arithmetically definable theory.

Theorem 2 (Kikuchi and Kurahashi)

If T is Σ_{n+1}-definable and Σ_n-sound extension of **PA**, *then T is not Π_{n+1}-decisive.*

The generalization of G1 to arithmetically definable theory

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The optimality of this generalization is shown by Salehi and Seraji.

Theorem 3 (Salehi and Seraji)

There exists a Σ_{n+1}-definable, Σ_{n-1}-sound ($n \geq 1$) and complete theory which contains **Q**.

So it is not true that any consistent extension of **PA** is incomplete.

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- This interpretation is false from Turing's work and Feferman's work.
- Turing's work shows that any true Π_1-sentence of arithmetic can be provable in some transfinite iteration of **PA**.
- Feferman's work extends Turing's work and shows that for any true sentence ϕ of arithmetic, there is a way to iterate the reflection principle and decide ϕ.

Three comments on G2

- **Con**(T) is not provable in T for any consistent R.E. extension T of **PA** in L(**PA**), but T is reflective: **Con**(S) is provable in T for any finite sub-theory S of T.
- From G2, we cannot get that **Con**(T) is independent of T: it is not enough to show that ¬**Con**(**PA**) is not provable in **PA** only assuming **PA** is consistent. But we could prove that **Con**(**PA**) is independent of **PA** by assuming that **PA** is 1-consistent.
- The notion of consistency is not absolute. A theory may be consistent from the external perspective but inconsistent from the internal perspective. For example, let $T = \mathbf{PA} + \neg\mathbf{Con}(\mathbf{PA})$. From G2, T is consistent from the external perspective. But since $T \vdash \neg\mathbf{Con}(T)$, T is not consistent from the internal perspective of T.

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- Gentzen constructed a theory **T** (primitive recursive arithmetic with the additional principle of quantifier-free transfinite induction up to the ordinal ϵ_0 where ϵ_0 is the first ordinal α such that $\omega^\alpha = \alpha$) and proved that **Con**(**PA**) is provable over **T**.
- Gentzen's theory **T** contains **Q** but does not contain **PA**.
- Gentzen's theory **T** interprets **PA**: by the arithmetized completeness theorem, $\mathbf{Q} + \mathbf{Con}(\mathbf{PA})$ interprets **PA**.
- But **PA** does not interpret Gentzen's theory **T**: by Pudlák's result, no consistent theory T that contains Robinson arithmetic **Q** can interpret $\mathbf{Q} + \mathbf{Con}(T)$.
- So $\mathbf{PA} \lhd \mathbf{T}$.

Provability and Truth

Definition 2

(1) **Prof**$=\{\ulcorner\phi\urcorner : \phi$ *is sentence and* **PA** $\vdash \phi\}$.

(2) **Truth**$=\{\ulcorner\phi\urcorner : \phi$ *is sentence and* $\mathfrak{N} \models \phi\}$.

Theorem 4 (Tarski)

Truth *is not definable in* $\mathfrak{N}$.

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Truth	**Prof**
not definable in $\mathfrak{N}$	definable in $\mathfrak{N}$
not arithmetic	recursive eumerable
not recursive	not recursive
not representable in **PA**	not representable in **PA**

Solovay's arithmetical completeness theorem

- An arithmetic interpretation is a function that assigns to each formula of modal logic a sentence of the language of arithmetic.
- **GL** is a modal system consisting of the schemes of axiom of **K** and $\Box(\Box A \to A) \to \Box A$.
- **GLS** is a modal system consisting of all theorems of **GL** and instances of the scheme of axiom $\Box A \to A$ with only one inference rule the modus ponens.

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Theorem 5 (Solovay)

Arithmetical completeness theorem for **GL** *For any modal formula* ϕ, $\mathbf{GL} \vdash \phi$ *iff* $\mathbf{PA} \vdash \phi^f$ *for every arithmetic interpretation* f.

Arithmetical completeness theorem for **GLS** *For any modal formula* ϕ, $\mathbf{GLS} \vdash \phi$ *iff* $\mathfrak{N} \models \phi^f$ *for every arithmetic interpretation* f.

Different proofs of incompleteness

- Constructive proof: directly construct the independence sentence
- Proof via the Diagonalization Lemma
- Proof via logical paradox
- Proof via recursion theory
- Proof via model theory

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Question 1

Could we give a self-reference-free proof of Gödel's incompleteness theorem?

Incompleteness and logical paradox

- Incompleteness is closely related to paradox.
- Gödel claimed: "Any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions".
- In Gödel's proof of G1, he imitated the Liar Paradox to construct a self-reference sentence called Gödel's sentence **G** which says that **G** is not provable in **PA**.

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Different proofs of incompleteness theorems via paradox:

Gödel	Liar Paradox
Boolos	Berry's paradox
Kurahashi	Yablo's Paradox
Kritchman	Unexpected Examination Paradox
Cieśliński	Grelling-Nelson Paradox

Mathematical examples of **G1** for **PA**

Question 2

Could we find a sentence about arithmetic with interesting mathematical contents which is independent of **PA***?*

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The following are examples of mathematically natural independent statements of **PA**.

Paris-Harrington Paris-Harrington principle

Kirby and Paris The Goodstein sequence

Kirby and Paris The Hercules-Hydra game

Kanamori-McAloon The Kanamori-McAloon principle

Beklemishev The Worm principle

Properties of mathematical examples

- All these naturally combinatorial independent principles with real mathematical contents are in fact provably equivalent in **PA** to a certain metamathematical sentence.
- Let $Rfn_{\Sigma_1}(\mathbf{PA})$ denote the sentence which expresses the reflection principle for Σ_1 sentences. McAloon proved that $\mathbf{PA} \vdash \varphi \leftrightarrow Rfn_{\Sigma_1}(\mathbf{PA})$, where φ is any one of the above independent principles.
- These independent principles are provable in some fragments of second order arithmetic but are more complex than Gödel's sentence: Gödel's sentence is equivalent to $\mathbf{Con}(\mathbf{PA})$ in **PA**; but all these principles are not only independent of **PA** but also independent of $\mathbf{PA} + \mathbf{Con}(\mathbf{PA})$.

Provability predicate

- Let T be a recursively axiomatized consistent extension of **PA**. We say that G2 holds for T if the consistency statement of T is not provable in T.
- The key question is: how to eliminate the vagueness in the consistency statement in a precise way.
- Whether G2 holds for T depends on the definition of provability predicate.

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- Whether G2 holds for T depends on the definition of provability predicate.

We say that a formula $\mathbf{Prf}_T(x, y)$ is a proof predicate of T iff it is Δ_1 and satisfies the following conditions:

(1) For any formula ϕ, $T \vdash \phi$ iff $T \vdash \mathbf{Prf}_T(\ulcorner \phi \urcorner, \overline{n})$ for some natural number n;

(2) $T \vdash \forall x(\exists y \, \mathbf{Prf}_T(x, y) \rightarrow \forall z \exists s > z \mathbf{Prf}_T(x, s))$;

(3) $T \vdash \forall y(\exists x \, \mathbf{Prf}_T(x, y) \rightarrow \exists! x \, \mathbf{Prf}_T(x, y))$.

Derivability conditions

- Define the provability predicate $\mathbf{Pr}_T(x)$ as $\mathbf{Pr}_T(x) \triangleq \exists y\, \mathbf{Prf}_T(x, y)$.
- Define the consistency statement $\mathbf{Con}(T)$ as $\neg\mathbf{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

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Definition 3 (Drivability conditions)

D1 *If* $T \vdash \varphi$, *then* $T \vdash \mathbf{Pr}_T(\ulcorner\varphi\urcorner)$*;*

D2 $T \vdash \mathbf{Pr}_T(\ulcorner\varphi\urcorner) \to (\mathbf{Pr}_T(\ulcorner\varphi \to \psi\urcorner) \to \mathbf{Pr}_T(\ulcorner\psi\urcorner))$*;*

D3 $T \vdash \mathbf{Pr}_T(\ulcorner\varphi\urcorner) \to \mathbf{Pr}_T(\ulcorner\mathbf{Pr}_T(\ulcorner\varphi\urcorner)\urcorner)$*.*

G2 holds for the standard provability predicate

- The derivability condition **D1** holds for any provability predicate.

Definition 4

We say a provability predicate is standard if it satisfies conditions **D2** *and* **D3**.

Theorem 6 (Gödel)

Let T be any recursively enumerable consistent extension of **PA**. *If* $\mathbf{Pr}_T(x)$ *is a standard provability predicate, then* $T \nvdash \mathbf{Con}(T)$.

G2 fails for the Rosser provability predicate

- Being a consistency statement is not an absolute concept but a role w.r.t. a choice of the provability predicate.
- Define the Rosser provability predicate $\mathbf{Pr}^R_{\mathbf{PA}}(x)$ as the formula $\exists y(\mathbf{Prf}_{\mathbf{PA}}(x, y) \wedge \forall z \leq y \neg \mathbf{Prf}_{\mathbf{PA}}(\dot{\neg}(x), z))$, where $\dot{\neg}$ is a function symbol expressing a primitive recursive function calculating the code of $\neg\phi$ from the code of ϕ.
- However, $\mathbf{Con}^R(\mathbf{PA}) \triangleq \neg \mathbf{Pr}^R_{\mathbf{PA}}(\ulcorner 0 = 1 \urcorner)$ is provable in **PA**.
- So the Rosser provability predicate at least does not satisfy one of the conditions **D2** and **D3**.

A general definition of provability predicate

Now we give a more general definition of provability predicate for T.

Definition 5

Let T be any recursively axiomatized consistent extension of **PA** *and $\alpha(x)$ be a formula in $L(T)$.*

(1) *Define the formula $\mathbf{Prf}_\alpha(x, y)$ saying "y is the Gödel number of a proof of the formula with Gödel number x from the set of all sentences satisfying $\alpha(x)$".*

(2) *Define the provability predicate $\mathbf{Pr}_\alpha(x)$ of $\alpha(x)$ as $\mathbf{Pr}_\alpha(x) \triangleq \exists y \mathbf{Prf}_\alpha(x, y)$ and consistency statement $\mathbf{Con}_\alpha(T)$ as $\triangleq \neg \mathbf{Pr}_\alpha(\perp)$.*

Properties of provability predicate under numerations

Let T be any recursively axiomatized consistent extension of **PA** and $\alpha(x)$ be a formula in $L(T)$.

Definition 6

$\alpha(x)$ is a numeration of T if for any n, $\mathbf{PA} \vdash \alpha(\overline{n})$ iff n is the Gödel number of some $\phi \in T$.

If $\alpha(x)$ is a numeration of T, then $\mathbf{Pr}_\alpha(x)$ satisfies the following properties:

- **D1′** If $T \vdash \varphi$, then $\mathbf{PA} \vdash \mathbf{Pr}_\alpha(\ulcorner \varphi \urcorner)$;
- **D2′** $\mathbf{PA} \vdash \mathbf{Pr}_\alpha(\ulcorner \varphi \urcorner) \rightarrow (\mathbf{Pr}_\alpha(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow \mathbf{Pr}_\alpha(\ulcorner \psi \urcorner))$;
- **D3′** If φ is Σ_1, then $\mathbf{PA} \vdash \varphi \rightarrow \mathbf{Pr}_\alpha(\ulcorner \varphi \urcorner)$.

The intensionality of G2 for **PA**

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Now we give a new reformulation of G2 via numerations.

Theorem 7 (Gödel)

Let T be any recursively enumerable consistent extension of **PA**. *If $\alpha(x)$ is any Σ_1 numeration of T, then $T \nvdash \mathbf{Con}_\alpha(T)$.*

- The intensionality of G2 says that whether G2 holds for **PA** depends on the numeration of **PA**.
- G2 holds for Σ_1 numerations of **PA**, but fails for some Π_1 numerations of **PA**: Feferman showed that if T is a R.E. extension of **PA**, then there is a Π_1 numeration $\tau(u)$ of T such that $T \vdash \mathbf{Con}_\tau(T)$.

Characterizing the consistency statement

- G2 is not coordinate-free (dependent on the numerations of **PA**).
- One way to eliminate the intensionability of G2 is to uniquely characterize the consistency statement in some sense.
- Consistency for finitely axiomatized sequential theories can be uniquely characterized modulo EA-provable equivalence.
- But characterizing the consistency of infinitely axiomatized R.E. theories is more delicate and a big open problem in the current research on the intensionability of G2.

Provability logic under numerations

- Provability logic provides us a new way to examine the intensionability of the provability predicate.
- Let T be any recursively axiomatized consistent extension of **PA** and $\alpha(x)$ be a numeration of T. The provability logic $\mathbf{PL}_\alpha(T)$ is the set of all modal principles which are verifiable in T when the modal operator $\Box$ is interpreted as $\mathbf{Pr}_\alpha(x)$.

Theorem 8 (Solovay's arithmetical completeness theorem)

Let T be any recursively axiomatized consistent extension of **PA**. *If T is Σ_1-sound, then for any Σ_1 numeration $\alpha(x)$ of T, the provability logic $\mathbf{PL}_\alpha(T)$ is precisely* **GL**.

The classification of provability logic

Let T be a recursively axiomatized consistent extension of **PA**.

Question 3

Which normal modal logic is a provability logic $\mathbf{PL}_\tau(T)$ *of some* Σ_n *numeration* $\tau(x)$ *of* T*?*

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Theorem 9 (Kurahashi, 2018)

(1) *There exists a* Σ_2 *numeration* $\alpha(x)$ *of* T *such that the provability logic* $\mathbf{PL}_\alpha(T)$ *is* $\mathbf{K}$.

(2) *For each* $n \geq 2$*, there exists a* Σ_2 *numeration* $\tau(x)$ *of* T *such that the provability logic* $\mathbf{PL}_\tau(T)$ *coincides with modal logic* $\mathbf{K} + \Box(\Box^n p \to p) \to \Box p$.

(3) *There exists a Rosser provability predicate whose provability logic is exactly the normal modal logic* $\mathbf{KD}$ *where* $\mathbf{KD} = \mathbf{K} + \neg\Box\bot$.

Definition of the higher order arithmetic

Definition 7

(1) $\mathsf{Z}_2 = \mathsf{ZFC}^- +$ *Every set is countable where* ZFC^- *denotes* ZFC *with the Power Set Axiom deleted and Collection instead of Replacement.*

(2) $\mathsf{Z}_3 = \mathsf{ZFC}^- + \mathcal{P}(\omega)$ *exists* $+$ *Every set is of cardinality* $\leq |\mathcal{P}(\omega)|$.

(3) $\mathsf{Z}_4 = \mathsf{ZFC}^- + \mathcal{P}(\mathcal{P}(\omega))$ *exists* $+$ *Every set is of cardinality* $\leq |\mathcal{P}(\mathcal{P}(\omega))|$.

Definition of the higher order arithmetic

Definition 7

(1) $Z_2 = ZFC^- +$ *Every set is countable where* ZFC^- *denotes* ZFC *with the Power Set Axiom deleted and Collection instead of Replacement.*

(2) $Z_3 = ZFC^- + \mathcal{P}(\omega)$ *exists* $+$ *Every set is of cardinality* $\leq |\mathcal{P}(\omega)|$.

(3) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ *exists* $+$ *Every set is of cardinality* $\leq |\mathcal{P}(\mathcal{P}(\omega))|$.

Corollary 1

If Z_2 *is consistent, then there is a true sentence about analysis which is not provable in* Z_2.

Incompleteness for the high order arithmetic

Fact 1

Many classic mathematical theorems about analysis which are expressible in Z_2 *are provable in* Z_2*.*

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Relativized Hilbert's program to Z_2 *Is Z_2 complete for classic mathematical theorems expressible in Z_2?*

Motivation Finding a counterexample for this question which is expressible in Z_2 but not provable in Z_2.

Harrington's Principle

Harrington's theorem $Det(\Sigma^1_1)$ implies $0^\sharp$ exists.

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Harrington's proof of "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" in ZF is done in two steps:

First Step $Det(\Sigma^1_1)$ implies HP;

Second Step HP implies $0^\sharp$ exists.

The counterexample

The first step "$Det(\Sigma^1_1)$ implies HP" is provable in Z_2.

A natural question: is "HP implies $0^\sharp$ exists" provable in Z_2?

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Theorem 10 (Y.Cheng, Phd thesis)

(1) *"HP implies $0^\sharp$ exists" is expressible in Z_2 but not provable in Z_2;*

(2) *"HP implies $0^\sharp$ exists" is not provable in Z_3;*

(3) *"HP implies $0^\sharp$ exists" is provable in Z_4;*

(4) *So Z_4 is the minimal system in higher order arithmetic to show that "HP implies $0^\sharp$ exists".*

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Theorem 11 (Y.Cheng, Ralf Schindler)

(1) $Z_2 + \mathrm{HP}$ *is equiconsistent with* ZFC.

(2) $Z_3 + \mathrm{HP}$ *is equiconsistent with* $\mathrm{ZFC} +$ *there exists a remarkable cardinal.*

Part Two: Finding the limit of incompleteness

Finding the limit of incompleteness

Question 5 (The big open question)

Exactly how much information of **PA** *is needed for the proof of* G1*?*

Let T be a recursively axiomatizable consistent theory.

Definition 9

G1 *holds for* T *iff for any recursively axiomatizable consistent theory* S*, if* T *is interpretable in* S*, then* S *is incomplete.*

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Finding the limit of incompleteness Could we find a minimal theory S with respect to interpretation such that G1 holds for S?

Definition 10

(1) *T is essentially undecidable iff any recursively axiomatizable consistent extension of T is undecidable.*

(2) *T is essentially incomplete iff any recursively axiomatizable consistent extension of T is incomplete.*

Proposition 1

Let T be a recursively axiomatizable consistent theory. The followings are equivalent:

(1) G1 *holds for T.*

(2) *T is essentially undecidable.*

(3) *T is essentially incomplete.*

In the following, I will review some essentially undecidable theories weaker than **PA** w.r.t. interpretation in the literature.

Robinson's **Q**

Definition 11

Robinson arithmetic **Q** *is defined in the language* $\{\mathbf{0}, \mathbf{S}, +, \cdot\}$ *with the following axioms:*

Q1 $\forall x \forall y(\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$;

Q2 $\forall x(\mathbf{S}x \neq \mathbf{0})$;

Q3 $\forall x(x \neq \mathbf{0} \rightarrow \exists y(x = \mathbf{S}y))$;

Q4 $\forall x \forall y(x + \mathbf{0} = x)$;

Q5 $\forall x \forall y(x + \mathbf{S}y = \mathbf{S}(x + y))$;

Q6 $\forall x(x \cdot \mathbf{0} = \mathbf{0})$;

Q7 $\forall x \forall y(x \cdot \mathbf{S}y = x \cdot y + x)$.

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- Q4 $\forall x \forall y(x + \mathbf{0} = x)$;
- Q5 $\forall x \forall y(x + \mathbf{S}y = \mathbf{S}(x + y))$;
- Q6 $\forall x(x \cdot \mathbf{0} = \mathbf{0})$;
- Q7 $\forall x \forall y(x \cdot \mathbf{S}y = x \cdot y + x)$.

PA consists of axioms $Q1$-$Q2$, $Q4$-$Q7$ and the following axiom scheme of induction:
$(\phi(\mathbf{0}) \land \forall x(\phi(x) \to \phi(\mathbf{S}x))) \to \forall x\, \phi(x)$ where ϕ is a formula with at least one free variable x.

Properties of Robinson arithmetic **Q**

- Robinson arithmetic **Q** is very weak and inadequate to formalize arithmetic: for instance, **Q** does not even prove that addition is associative.
- Robinson shows that any theory that interprets **Q** is undecidable and hence **Q** is essentially undecidable.
- **Q** is minimal essentially undecidable in the sense that: if deleting any axiom of **Q**, then the remaining theory is not essentially undecidable and has a complete decidable extension.
- **Q** represents a rich degree of interpretability since a lot of stronger theories are interpretable in it.

Fragments of **PA** extending **Q**

Definition 12

$I\Sigma_n$ is $\mathbf{Q}$ plus induction for Σ_n formulas and $B\Sigma_{n+1}$ is $I\Sigma_0$ plus collection for Σ_{n+1} formulas.

It is well known that the theories form a strictly increasing hierarchy:

$$I\Sigma_0, B\Sigma_1, I\Sigma_1, B\Sigma_2, \cdots I\Sigma_n, B\Sigma_{n+1}, \cdots, \mathbf{PA}.$$

Definition 13

(1) *Define $\omega_1(x) = x^{|x|}$ and $\omega_{n+1}(x) = 2^{\omega_n(|x|)}$ where $|x|$ is the length of the binary expression of x.*

(2) *Let Ω_n denote the statement $\forall x \exists y(\omega_n(x) = y)$ which says that $\omega_n(x)$ is total.*

The hierarchy theorem

Theorem 12 (Petr Hájek, Fernando Ferreira and Gilad Ferreira)

(1) *For any* $n \geq 1$, $I\Sigma_0 + \Omega_n$ *is interpretable in* $\mathbf{Q}$.

(2) $I\Sigma_{n+1}$ *is not interpretable in* $B\Sigma_{n+1}$.

(3) *For each* $n \geq 1$, $B\Sigma_1 + \Omega_n$ *is interpretable in* $I\Sigma_0 + \Omega_n$.

(4) *For each* $n \geq 0$, $B\Sigma_{n+1}$ *is interpretable in* $I\Sigma_n$.

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(3) *For each* $n \geq 1$, $B\Sigma_1 + \Omega_n$ *is interpretable in* $I\Sigma_0 + \Omega_n$.
(4) *For each* $n \geq 0$, $B\Sigma_{n+1}$ *is interpretable in* $I\Sigma_n$.

Corollary 2

(1) *The following theories are all mutually interpretable:*
- $\mathbf{Q}$
- $I\Sigma_0$
- $I\Sigma_0 + \Omega_n (n \geq 1)$
- $B\Sigma_1$
- $B\Sigma_1 + \Omega_n (n \geq 1)$

(2) *For* $n \geq 1$, $I\Sigma_n$ *and* $B\Sigma_{n+1}$ *are mutually interpretable;*
(3) $\mathbf{Q} \lhd I\Sigma_0 + exp \lhd I\Sigma_1 \lhd I\Sigma_2 \lhd \cdots \lhd I\Sigma_n \lhd \cdots \lhd \mathbf{PA}$.

Variants of **Q**: $\mathbf{Q}^-$

Andrzej Grzegorczyk considered a theory $\mathbf{Q}^-$ in which addition and multiplication are possibly non-total functions. The language of $\mathbf{Q}^-$ is $\{\mathbf{0}, \mathbf{S}, A, M\}$ where A and M are ternary relations, and the axioms of $\mathbf{Q}^-$ are the axioms $Q1$-$Q3$ of **Q** plus the following six axioms about A and M:

- A $\forall x \forall y \forall z_1 \forall z_2 (A(x, y, z_1) \wedge A(x, y, z_2) \rightarrow z_1 = z_2)$;
- M $\forall x \forall y \forall z_1 \forall z_2 (M(x, y, z_1) \wedge M(x, y, z_2) \rightarrow z_1 = z_2)$;
- G4 $\forall x\, A(x, 0, x)$;
- G5 $\forall x \forall y \forall z (\exists u (A(x, y, u) \wedge z = S(u)) \rightarrow A(x, S(y), z))$;
- G6 $\forall x\, M(x, 0, 0)$;
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Theorem 13 (Švejdar)

$\mathbf{Q}$ is interpretable in $\mathbf{Q}^-$ and hence $\mathbf{Q}^-$ is essentially undecidable.

Variants of $\mathbf{Q}$: $\mathbf{Q}^+$

Let $\mathbf{Q}^+$ be the extension of $\mathbf{Q}$ with the following extra axioms (the language $L(\mathbf{Q}^+)$ extends $L(\mathbf{Q})$ and includes the binary relation symbol $\leq$):

- Q8 $(x + y) + z = x + (y + z)$;
- Q9 $x \cdot (y + z) = x \cdot y + x \cdot z$;
- Q10 $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
- Q11 $x + y = y + x$;
- Q12 $x \cdot y = y \cdot x$;
- Q13 $x \leq y \leftrightarrow \exists z(x + z = y)$.

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Theorem 14 (Fernando Ferreira and Gilad Ferreira)

$\mathbf{Q}^+$ *is interpretable in* $\mathbf{Q}$.

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The theory **PA**$^-$ has the following axioms with the language $L(\mathbf{PA}^-) = L(\mathbf{PA}^-) \cup \{\leq\}$:

(1) $x + 0 = x$;

(2) $x + y = y + x$;

(3) $(x + y) + z = x + (y + z)$;

(4) $x \cdot 1 = x$;

(5) $x \cdot y = y \cdot x$;

(6) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

(7) $x \cdot (y + z) = x \cdot y + x \cdot z$;

(8) $x \leq y \vee y \leq x$;

(9) $(x \cdot y \wedge y \cdot z) \to x \leq z$;

(10) $x + 1 \not\leq x$;

(11) $x \leq y \to (x = y \vee x + 1 \leq y)$;

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Fact 2

PA$^-$ *is interpretable in* **Q**.

The theory of concatenation **TC**

Definition 14

(A. Grzegorczyk) The theory of concatenation **TC** *has the language* $\{\frown, \alpha, \beta\}$ *and the following axioms:*

- TC1 $\forall x \forall y \forall z (x \frown (y \frown z) = (x \frown y) \frown z)$;
- TC2 $\forall x \forall y \forall u \forall v (x \frown y = u \frown v \to ((x = u \wedge y = v) \vee \exists w ((u = x \frown w \wedge w \frown v = y) \vee (x = u \frown w \wedge w \frown y = v))))$;
- TC3 $\forall x \forall y (\alpha \neq x \frown y)$;
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Theorem 15 (Švejdar, M. Ganea, Visser)

Q *is interpretable in* **TC**.

$\mathbf{S}^1_2$ is a finitely axiomatizable theory introduced by Buss. The following theories are all essentially undecidable and mutually interpretable:

$$\mathbf{TC}, \mathbf{Q}^-, \mathbf{Q}^+, \mathbf{PA}^-, \mathbf{S}^1_2, \mathbf{Q}.$$

The system $\mathbf{R}$

In the following, the working language is $\{\overline{0}, \cdots, \overline{n} \cdots, +, \cdot, \leq\}$ with infinitely many constants as names for natural numbers and with $\leq$ as primitive symbol.

Definition 15 (Tarski, Mostowski and Robinson)

The system $\mathbf{R}$

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Definition 15 (Tarski, Mostowski and Robinson)

(1) *Let* $\mathbf{R}$ *be the system consisting of schemes* $Ax1$-$Ax5$ *where* $m, n \in \mathbb{N}$.

$Ax1$ $\overline{m} + \overline{n} = \overline{m+n}$;

$Ax2$ $\overline{m} \neq \overline{n}$ *if* $m \neq n$;

$Ax3$ $\overline{m} \cdot \overline{n} = \overline{m \cdot n}$;

$Ax4$ $\forall x(x \leq \overline{n} \rightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$;

$Ax5$ $\forall x(x \leq \overline{n} \vee \overline{n} \leq x)$;

$Ax4'$ $\forall x(x \leq \overline{n} \leftrightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$.

(2) *Let* $\mathbf{R}_0$ *be the system consisting of schemes* $Ax1$-$Ax4$. *Let* $\mathbf{R}_1$ *be the system consisting of schemes* $Ax1$-$Ax3$ *and* $Ax4'$. *Let* $\mathbf{R}_2$ *be the system consisting of schemes* $Ax2$-$Ax3$ *and* $Ax4'$.

Properties of **R**

(1) **R** is not finitely axiomatizable but is locally finitely satisfiable (any finite sub-theory of T has a finite model).

(2) **R** $\lhd$ **Q** since **Q** is not interpretable in **R**.

(3) All recursive functions are representable in **R** and hence **R** is essentially undecidable.

(4) (Cobham) Any R.E. theory that weakly interprets **R** is undecidable (U weakly interprets a theory V if V is interpretable in some consistent extension $\overline{U}$ of U in the same language).

(5) **R** is not minimal essentially undecidable.

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(5) **R** is not minimal essentially undecidable.

Theorem 16 (Albert Visser)

Suppose T is an R.E. theory. Then T is locally finitely satisfiable iff T is interpretable in **R**.

Properties of variants of **R**

- **R** is not minimal essentially undecidable.
- $\mathbf{R}_0$ is also not essentially undecidable.
- $\mathbf{R}_1$ is essentially undecidable but not minimal essentially undecidable.
- $\mathbf{R}_2$ is essentially undecidable and minimal essentially undecidable.
- $\mathbf{R}, \mathbf{R}_0, \mathbf{R}_1$ and $\mathbf{R}_2$ are all essentially undecidable and mutually interpretable.

Summary

In a summary, we have:

(1) G1 holds for the following theories and they are mutually interpretable with **Q**:

$$I\Sigma_0, I\Sigma_0+\Omega_n (n \geq 1), B\Sigma_1, B\Sigma_1+\Omega_n (n \geq 1), \mathbf{TC}, \mathbf{Q}^-, \mathbf{Q}^+, \mathbf{PA}^-, \mathbf{S}^1_2.$$

(2) G1 holds for the following theories and they are mutually interpretable with **R**:

$$\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2.$$

Question 6

Could we find a theory S such that G1 *holds for S and $S \lhd \mathbf{R}$?*

In the following, I first answer this question positively based on JeŘábek's work by providing many examples for Question 6 via some model theory.

Some definitions from model theory

Definition 16

(1) *A model M is existentially closed if for every model $N \supseteq M$, we have $M \preceq N$: every existential formula with parameters from M which is satisfied in N is already satisfied in M.*

(2) *A consistent theory T is said to be model complete if for all models M, N of T, if $M \subseteq N$ then $M \preceq N$.*

(3) *T has quantifier elimination if every formula is equivalent in T to a quantifier-free formula.*

(4) *Theories T and S in the same language are companions if every model of T embeds in a model of S, and every model of S embeds in a model of T.*

Some facts from model theory

Definition 17

(1) *For any structure M, we denote by $Diag(M)$ its diagram: the set of quantifier-free sentences true in M in the language of M augmented by constants for any element of M.*

(2) *A model completion of a theory T is a model companion $\bar{T}$ of T such that for any $M \models T$, the theory $\bar{T} + Diag(M)$ is complete.*

Fact 3

(1) *Given a consistent theory T, the following are equivalent:*
   (1) *T is model complete;*
   (2) *All models $M \models T$ are essentially closed models of T;*
   (3) *Every formula is equivalent in T to an existential formula.*

(2) *If T is a $\forall\exists$-axiomatized theory, then every model $M \models T$ embeds in an essentially closed model of T.*

(3) *If T is an universal theory, a companion T^* of T is a model completion of T if T^* has quantifier elimination.*

Definition 18

([2]) Let L be a finite language, and Θ be a finite set of L-terms closed under subterms such that the variables in Θ are among $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}$. Let $\tau^R_{\bar{t}} \in \{0,1\}$ for every k-ary relation $R \in L$, and every $t_0, \ldots, t_{k-1} \in \Theta$. Let $\sim$ be an equivalence relation on Θ such that:

(i) *If $R \in L$ is k-ary, and $t_i \sim s_i$ for each $i < k$, then $\tau^R_{\bar{t}} = \tau^R_{\bar{s}}$.*

(ii) *If $F \in L$ is k-ary, and $t \triangleq F(\bar{t}) \in \Theta$ and $s \triangleq F(\bar{s}) \in \Theta$ satisfy $t_i \sim s_i$ for each $i < k$, then $t \sim s$.*

Then the elementary existential formula $\exists \bar{y} \theta_{\Theta,\sim,\tau}(\bar{x}, \bar{y})$ is defined by

$$\theta_{\Theta,\sim,\tau}(\bar{x}, \bar{y}) \triangleq \bigwedge_{t,s\in\Theta, t\sim s} t = s \wedge \bigwedge_{t,s\in\Theta, t\nsim s} t \neq s \wedge \bigwedge_{R\in L, \bar{t}\in\Theta} R^{\tau^R_{\bar{t}}}(\bar{t}), \tag{1}$$

where $\phi^1 \triangleq \phi$, $\phi^0 \triangleq \neg\phi$.

Definition 19

([2]) Let $L, \Theta, \bar{x}, \bar{y}, \sim, \tau$ be as in Definition 18. We define a subset $\Delta \subseteq \Theta$, and for each $t \in \Delta$ a term $t^*(\bar{x})$, as follows:

(i) *Every variable x_i is in Δ, and $x_i^* \triangleq x_i$.*

(ii) *If $t \sim s \in \Delta$, then $t \in \Delta$, and $t^* \triangleq s^*$.*

(iii) *If $t = F(t_0, ..., t_{k-1}) \in \Theta$, and $t_0, \cdots, t_{k-1} \in \Delta$, then $t \in \Delta$, and $t^* \triangleq F(t_0^*, \cdots, t_{k-1}^*)$.*

Define an open formula $\theta^*_{\Theta,\sim,\tau}(\bar{x})$ as

$$\bigwedge_{t,s \in \Delta, t \sim s} t^* = s^* \wedge \bigwedge_{t,s \in \Delta, t \nsim s} t^* \neq s^* \wedge \bigwedge_{R \in L, \bar{t} \in \Delta} R^{\tau^R_{\bar{t}}}(\bar{t}) \wedge \bigwedge_{t \triangleq F(\bar{t}) \in \Delta, \bar{t} \in \Delta} t^* = F(\bar{t}^*)$$

Definition 20

([2]) If L is a finite language, let EC_L denote the theory axiomatized by the formulas

$$\theta^*_{\Theta,\sim,\tau}(\bar{x}) \to \exists y\, \theta_{\Theta,\sim,\tau}(\bar{x}, y). \qquad (2)$$

for all $\Theta, \sim, \tau$ as in Definition 18 with $m = 1$. For infinite L, let $EC_L = \bigcup\{EC_{\bar{L}} : \bar{L} \subseteq L \text{ finite}\}$.

Properties of EC_L

Theorem 17 (Folklore)

For any language L, the followings hold:

(i) EC_L *has elimination of quantifiers;*

(ii) EC_L *is model complete;*

(iii) *Models of* EC_L *are exactly the existentially closed L-structures; in particular, every L-structure embeds in a model of* EC_L*;*

(iv) EC_L *is the companion of the empty L-theory (the theory with no extra-logical axioms);*

(v) EC_L *is the model completion of the empty L-theory.*

Definition 21

$\langle S, T\rangle$ is a recursively inseparable pair if S and T are disjoint recursively enumerable sets, and there is no recursive set $X \subseteq \mathbb{N}$ such that $S \subseteq X$ and $X \cap T = \emptyset$.

Definition 21

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Definition 22

Let $\langle A, B\rangle$ be a recursively inseparable pair. Let L be the finite language $\{\mathbf{0}, \mathbf{S}, \mathbf{P}\}$. Consider the following theory $U_{\langle A,B\rangle}$:

(1) $\overline{m} \neq \overline{n}$ *if* $m \neq n$;

(2) $\mathbf{P}(\overline{n})$ *if* $n \in A$;

(3) $\neg\mathbf{P}(\overline{n})$ *if* $n \in B$.

In the following, let $\langle A, B\rangle$ be an arbitrary recursively inseparable pair.

Lemma 1

G1 *holds for* $U_{\langle A,B\rangle}$.

Proof.

It suffices to show that $U_{\langle A,B\rangle}$ is essentially incomplete. □

Definition 23

$\phi(x)$ *numerates* A *in* T *if for any* n, $n \in A$ *iff* $T \vdash \phi(\bar{n})$.

Lemma 2

Let A *and* B *be disjoint recursively enumerable sets and* T *be an extension of* $\mathbf{Q}$. *Then there is a* Σ_1 *formula* $\phi(x)$ *such that* $\phi(x)$ *numerates* A *in* T *and* $\neg\phi(x)$ *numerates* B *in* T.

Lemma 3

$U_{\langle A,B\rangle}$ *is interpretable in* $\mathbf{R}$.

A relation $R \subseteq X^2$ is asymmetric if there are no $a, b \in X$ such that $R(a, b)$ and $R(b, a)$.

Theorem 18

(Emil JeŘábek, [2]) For any language L and formula $\phi(z, x, y)$, there is a constant n with the following property. Let $M \models EC_L$ and $u \in M$ be such that $M \models \exists x_0, \cdots, \exists x_{n-1} \bigwedge_{i<j<n} \phi(u, x_i, x_j)$. Then for every $m \in \omega$ and an asymmetric relation R on $\{0, \cdots, m-1\}$, $M \models \exists x_0, \cdots, \exists x_{m-1} \bigwedge_{\langle i,j \rangle \in R} \phi(u, x_i, x_j)$.

Proof.

- We may assume L contains no relation symbols and ϕ is open.
- Let p be the number of subterms of ϕ, and $k = 2^{256 \cdot p^2}$. Using Ramsey's theorem, let n be sufficiently large so that $n \to (7)^4_k$. I.e. for any function $f : [n]^4 \to k$, there exists a set $X \subseteq n$ with size 7 such that f is constant on $[X]^4$.

Proof: Continued

- Fix $M \models EC_L, u \in M$ and $\{a_i : i < n\} \subseteq M$ such that $M \models \phi(u, a_i, a_j)$ for $i < j < n$. In order to simplify the notation, we will assume u is given by a constant of L, and write just $\phi(x, y)$.
- Let Σ be the set of all subterms $t(x, y)$ of ϕ, and for every $i_0 < i_1 < i_2 < i_3 < n$, define

$$f(i_0, i_1, i_2, i_3) = \{j_0, j_1, j_2, j_3, t, s \in \mathbf{4}^4 \times S^2 : t^M(a_{i_{j_0}}, a_{i_{j_1}}) = s^M(a_{i_{j_2}}, a_{i_{j_3}})\}.$$

- f is a colouring of quadruples of numbers below n by at most k colours. Thus, we can find a 7-element homogeneous set $H \subseteq n$ for f. WLOG, we may assume $H = \{0, \cdots, 6\}$.
- Fix a set of variables $\{y_\alpha : \alpha < m\}$, and put $\Theta = \{t(y_\alpha, y_\beta) : \langle \alpha, \beta \rangle \in R, t(x, y) \in \Sigma\}$. If $t \in \Theta$, let $D(t)$ denote the set of $\alpha < m$ such that y_α occurs in t.

Proof: Continued

- A realization of t is an injective function $h : D(t) \to H$ such that if $\alpha, \beta \in D(t)$ and $\langle \alpha, \beta \rangle \in R$, then $h(\alpha) < h(\beta)$.
- If h is a realization of $t(y_\alpha, y_\beta)$, define $h(t) \in M$ as $t^M(a_{h(\alpha)}, a_{h(\beta)})$. A joint realization of a set of terms $\{t_0, \cdots, t_{l-1}\} \subseteq \Theta$ is an injective mapping $h : D(t_0) \cup \cdots D(t_{l-1}) \to H$ such that $h \restriction D(t_i)$ is a realization of t_i for any $i < l$.
- If $t, s \in \Theta$, and h is a joint realization of t and s, we define $t \sim s \Leftrightarrow h(t) = h(s)$.

Lemma 4

The relation $\sim$ is well defined and an equivalence relation on Θ which satisfies Definition 18.

- Let $\Delta \subseteq \Theta$ and $\{t^* : t \in \Delta\}$ be as in Definition 19 (for empty $\bar{x}$).
- The value of the closed term t^* in M coincides with $h(t)$ for any realization h of t. From this fact and the definition of $\sim$, we have $M \models \theta^*_{\Theta,\sim}$.
- Since M is existentially closed, we have $M \models \exists y_0, \cdots, y_{m-1} \theta_{\Theta,\sim}(\bar{y})$. Take $b_0, \cdots, b_{m-1} \in M$ such that $M \models \theta_{\Theta,\sim}(\bar{b})$.

Lemma 5

For $\langle \alpha, \beta \rangle \in R$, $i < j \in H$ and any sub-formula φ of ϕ, we have $M \models \varphi(b_\alpha, b_\beta) \Leftrightarrow M \models \varphi(a_i, a_j)$.

We have $M \models \bigwedge_{\langle \alpha,\beta \rangle \in R} \phi(b_\alpha, b_\beta)$.

The main theorem

Definition 24

Consider the following theory **T** *in the language* $\langle \in \rangle$ *axiomatized by the sentences*

$\exists z, x_0, \cdots, x_{n-1}(\bigwedge_{i<j<n} x_i \neq x_j \wedge \forall y(y \in z \leftrightarrow \bigvee_{i<n} y = x_i))$

for all $n \in \omega$.

The main theorem

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$\exists z, x_0, \cdots, x_{n-1}(\bigwedge_{i<j<n} x_i \neq x_j \wedge \forall y(y \in z \leftrightarrow \bigvee_{i<n} y = x_i))$
for all $n \in \omega$.

Theorem 19

For any recursively inseparable pair $\langle A, B \rangle$, *there is a theory* $U_{\langle A,B \rangle}$ *such that* G1 *holds for* $U_{\langle A,B \rangle}$ *and* $U_{\langle A,B \rangle} \lhd \mathbf{R}$.

- **T** is not weakly interpretable in EC_L for any language L: Suppose this does not hold and apply Theorem 18 to the formula which interprets $\bigwedge_{i<j<n} x_i \neq x_j \wedge \forall y(y \in z \leftrightarrow \bigvee_{i<n} y = x_i)$ and R a chain longer than n to get a contradiction.
- **R** is not weakly interpretable in EC_L for any language L.
- If **R** is interpretable in $U_{\langle A,B \rangle}$, then **R** is weakly interpretable in EC_L for some language L.
- **R** is not interpretable in $U_{\langle A,B \rangle}$.
- G1 holds for $U_{\langle A,B \rangle}$ and $U_{\langle A,B \rangle} \lhd \mathbf{R}$.

Question 7

(Visser) Would S with $S \trianglelefteq \mathbf{R}$ such that G1 *holds for S shares the universality property of $\mathbf{R}$ that every locally finitely satisfiable theory is interpretable in it.*

- The answer for this question is negative.
- We have shown that for any recursively inseparable pair $\langle A, B\rangle$, there is a theory $U_{\langle A,B\rangle}$ such that G1 holds for $U_{\langle A,B\rangle}$ and $U_{\langle A,B\rangle} \lhd \mathbf{R}$.
- The locally finitely satisfiable theory $\mathbf{T}$ is not interpretable in $U_{\langle A,B\rangle}$: if $\mathbf{T}$ is interpretable in $U_{\langle A,B\rangle}$, then $\mathbf{T}$ is weakly interpretable in EC_L for some language L.
- So for any recursively inseparable pair $\langle A, B\rangle$, the theory $U_{\langle A,B\rangle}$ is a counterexample for Visser's Question.

Questions for further research

Theorem 20 (Visser)

Suppose $\mathbf{R} \subseteq A$*, where* A *is finitely axiomatized and consistent. Then, there is a finitely axiomatized* B *such that* $\mathbf{R} \subseteq B \subseteq A$ *and* $B \lhd A$*.*

Define $X = \{S : \mathbf{R} \unlhd S \lhd \mathbf{Q}\}$. Theorem 20 shows that the structure $(X, \lhd)$ is not well founded w.r.t. finitely axiomatized theories.

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Define $X = \{S : \mathbf{R} \unlhd S \lhd \mathbf{Q}\}$. Theorem 20 shows that the structure $(X, \lhd)$ is not well founded w.r.t. finitely axiomatized theories.

Theorem 21 (Visser)

Suppose A *and* B *are finitely axiomatized theories that weakly interpret* $\mathbf{S^1_2}$*. Then there are finitely axiomatized theories* $\overline{A} \supseteq A$ *and* $\overline{B} \supseteq B$ *such that* $\overline{A}$ *and* $\overline{B}$ *are incomparable (i.e.* $\overline{A} \ntrianglelefteq \overline{B}$ *and* $\overline{B} \ntrianglelefteq \overline{A}$*).*

Theorem 21 shows that there are incomparable theories extending $\mathbf{Q}$.

Further Questions

Define $\mathsf{D} = \{\mathsf{S} : \mathsf{S} \lhd \mathbf{R}$ and G1 holds for $S\}$.

Question 8

(1) *Is* $(\mathsf{D}, \lhd)$ *well founded?*

(2) *Are any two elements of* $(\mathsf{D}, \lhd)$ *comparable?*

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Define $\mathsf{D} = \{\mathsf{S} : \mathsf{S} \lhd \mathbf{R} \text{ and G1 holds for } S\}$.

Question 8

(1) *Is* $(\mathsf{D}, \lhd)$ *well founded?*

(2) *Are any two elements of* $(\mathsf{D}, \lhd)$ *comparable?*

Conjecture 1

$(\mathsf{D}, \lhd)$ *is not well founded and has incomparable elements.*

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Thanks for your attention!