Robinson's theory R and a R-like Globaliser for c.e. theories

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Robinson's theory **R**

Robinson's theory **R** is introduced by Tarski, Mostowski and Robinson in [6].

Definition (Tarski, Mostowski and Robinson)

Robinson's theory **R** *consists of schemes* Ax1-Ax5 *in the language* $\{\mathbf{0}, \mathbf{S}, +, \times, \leq\}$ *where* $\leq$ *is a primitive binary relation symbol and* $\overline{n} = \mathbf{S}^n\mathbf{0}$ *for* $n \in \mathbb{N}$*:*

- Ax1 $\overline{m} + \overline{n} = \overline{m+n}$;
- Ax2 $\overline{m} \times \overline{n} = \overline{m \times n}$;
- Ax3 $\overline{m} \neq \overline{n}$, if $m \neq n$;
- Ax4 $\forall x(x \leq \overline{n} \rightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$;
- Ax5 $\forall x(x \leq \overline{n} \vee \overline{n} \leq x)$.

The notion of interpretation

The notion of interpretation is originally introduced by Tarski, Mostowski and Robinson in [6], that provides us with a method for comparing the strength of theories in different languages.

Definition (Notation of Interpretation)

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be countable first-order languages. Let U be an $\mathcal{L}_1$-theory and T be an $\mathcal{L}_2$-theory.

(1) *We write $T \trianglelefteq U$, if T is interpretable in U (i.e. there exists an interpretation of T in U).*

(2) *We write $T \lhd U$ if T is interpretable in U, but U is not interpretable in T.*

(3) *We say that S and T are mutually interpretable if $S \trianglelefteq T$ and $T \trianglelefteq S$.*

Some meta-mathematical properties of $\mathbf{R}$

Theorem (Forklore, many authors)

(1) *For each formula $\phi(x)$, there exists a sentence θ satisfying $\mathbf{R} \vdash \theta \leftrightarrow \phi(\ulcorner \theta \urcorner)$.*

(2) *All recursive functions are representable in $\mathbf{R}$.*

(3) $\mathbf{R}$ *is essentially undecidable.*

(4) *Any r.e. theory that interprets $\mathbf{R}$ is undecidable.*

(5) *Σ_1-completeness holds for $\mathbf{R}$: for any Σ_1 sentence ϕ, $\mathfrak{N} \models \phi$ iff $\mathbf{R} \vdash \phi$.*

A definition

Definition

Let T be a recursively axiomatizable consistent theory. We say that G1 *holds for T if for any recursively axiomatizable consistent theory S, if T is interpretable in S, then S is incomplete.*

Proposition

Let T be a recursively axiomatizable consistent theory. The following are equivalent:

(1) G1 *holds for T.*

(2) *T is essentially incomplete.*

(3) *T is essentially undecidable.*

G1 for weak arithmeitc

Question

Is there a theory T such that G1 holds for T and $T \lhd \mathbf{R}$?

Theorem

For any recursively inseparable pair $\langle A, B\rangle$, there is a r.e. theory $U_{\langle A,B\rangle}$ such that $U_{\langle A,B\rangle} \lhd \mathbf{R}$ and G1 holds for $U_{\langle A,B\rangle}$.

Theorem (Visser, [7])

Suppose $\mathbf{R} \subseteq A$, where A is finitely axiomatized and consistent. Then, there is a finitely axiomatized B such that $\mathbf{R} \subseteq B \subseteq A$ and $B \lhd A$.

This theorem shows that the structure $\langle\{S : \mathbf{R} \unlhd S \lhd \mathbf{Q}\}, \lhd\rangle$ is not well founded w.r.t. finitely axiomatized theories.

The limit of theories below **R** for which G1 holds

Define $\mathrm{D} = \{S : S \lhd \mathbf{R}$ and G1 holds for theory $S\}$.

Question (Open)

Is $\langle \mathrm{D}, \lhd \rangle$ *well founded (or is it that for any* $S \in \mathrm{D}$*, there is* $T \in \mathrm{D}$ *such that* $T \lhd S$*)?*

Define $\overline{\mathrm{D}} = \{S : S <_T \mathbf{R}$ and G1 holds for theory $S\}$.

Theorem

$\overline{\mathrm{D}}$ *is not well founded: for any* $U \in \overline{\mathrm{D}}$*, there is* $V \in \overline{\mathrm{D}}$ *such that* $V <_T U$*. I.e. there is no minimal theory below* **R** *w.r.t. Turing reducibility such that* G1 *holds for it.*

G2 for weak arithmetic

Definition

For a consistent theory T, we say G2 *holds for T if $T \nvdash \mathbf{Con}(T)$.*

Now, we examine whether G2 holds for weak arithmetics.

Theorem (Pudlák)

There is no consistent r.e. theory S such that $(\mathbf{Q} + \mathbf{Con}(S)) \trianglelefteq S$.

As a corollary, G2 holds for any consistent r.e. theory interpreting $\mathbf{Q}$.

Question

Does G2 *hold for* $\mathbf{R}$*?*

The MRDP theorem

Definition (Folklore)

(1) *A Diophantine equation is one of the form* $P_1(\vec{x}) = P_2(\vec{x})$ *where* P_i *are polynomials over* $\mathbb{N} \setminus \{0\}$.

(2) *We say that a sentence is Diophantine iff it is a block of existential quantifiers followed by a Diophantine equation. I.e. a Diophantine sentence claims that a Diophantine equation has a solution.*

Theorem (The MRDP theorem)

Every Σ_1^0 *sentence is equivalent to some Diophantine sentence.*

G2 for **R**

- Since **Con**(**R**) is a Π^0_1 sentence, by the MRDP theorem, **Con**(**R**) is equivalent to $\forall x(P_1(x) \neq P_2(x))$ for some polynomials P_1 and P_2 over $\mathbb{N} \setminus \{0\}$: i.e. some Diophantine equation has no solution.
- In the following, we equate **Con**(**R**) with $\forall x(P_1(x) \neq P_2(x))$.

Theorem

$\mathbf{R} \nvdash \mathbf{Con}(\mathbf{R})$.

Proof.

- To show that $\mathbf{R} \nvdash \mathbf{Con}(\mathbf{R})$, it suffices to construct a model $\mathfrak{M}$ of **R** such that $\mathfrak{M} \models \exists x(P_1(x) = P_2(x))$.
- Define the model $\mathfrak{M} = \langle \mathbb{N} \cup \{\infty\}, \mathbf{0}, \mathbf{S}, +, \times, \leq \rangle$ as follows:
  (1) Define $\mathbf{S}\infty = \infty$; $x + \infty = \infty, \infty + x = \infty$ for all $x \in \mathbb{N} \cup \{\infty\}$;
  (2) $x \times \infty = \infty, \infty \times x = \infty$ for all $x \in \mathbb{N} \cup \{\infty\}$;
  (3) $x \leq \infty$ for any $x \in \mathbb{N} \cup \{\infty\}$.
- It is easy to check that $\mathfrak{M} \models \mathbf{R}$, but there is $\infty \in \mathfrak{M}$ such that $P_1(\infty) = P_2(\infty)$.

Visser's characterization of **R**

Definition (Folklore)

We say a theory T is locally finitely satisfiable if any its finite sub-theory T' has a finite model.

Visser's following theorem provides a characterization of r.e. theory T such that $T \trianglelefteq \mathbf{R}$.

Theorem (Visser, 2009)

For any r.e. theory T, T is locally finitely satisfiable if and only if T is interpretable in $\mathbf{R}$.

The following section is a joint work with Fedor Pakhomov.

Interpretations of models

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be countable first-order languages. Suppose U is a $\mathcal{L}_1$-theory, $\mathfrak{N}$ is a $\mathcal{L}_1$-model and $\mathfrak{M}$ is a $\mathcal{L}_2$-model.

Definition (Interpretations of models)

An interpretation τ of U in $\mathfrak{M}$ consists of

(1) *$\mathfrak{M}$-definable set $D_\tau \subseteq |\mathfrak{M}|$ (the domain of interpretation),*

(2) *$\mathfrak{M}$-definable set $P^\tau \subseteq (D_\tau)^n$, for each n-ary predicate symbol P from $\mathrm{Sgn}(U)$,*

(3) *$\mathfrak{M}$-definable function $f^\tau : (D_\tau)^n \to D_\tau$, for each n-ary function symbol f from $\mathrm{Sgn}(U)$,*

such that

$$(D_\tau, \langle P^\tau | P \in \mathrm{Sgn}(U)\rangle, \langle f^\tau | f \in \mathrm{Sgn}(U)\rangle) \models U.$$

- *We write $\tau : U \trianglelefteq \mathfrak{M}$ to indicate that τ is an interpretation of U in $\mathfrak{M}$.*
- *Similarly, we can define that τ is an interpretation of $\mathfrak{N}$ in $\mathfrak{M}$ if $(D_\tau, \langle P^\tau | P \in \mathrm{Sgn}(\mathfrak{N})\rangle, \langle f^\tau | f \in \mathrm{Sgn}(\mathfrak{N})\rangle) \cong \mathfrak{N}$.*

Different notions of interpretations

We require D_τ to be a definable set in $\mathfrak{M}$ in the above Definition. Depending on what exactly we consider to be a definable set we will get different notions of interpretations.

- τ is one-dimensional if $D_\tau \subseteq |\mathfrak{M}|$;
- τ is multi-dimensional if $D_\tau \subseteq |\mathfrak{M}|^n$ for some $n \in \omega$;
- τ is piece-wise multi-dimensional if D_τ is a disjoint union $A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ of definable sets $A_1 \subseteq |\mathfrak{M}|^{i_1}, \cdots, A_n \subseteq |\mathfrak{M}|^{i_n}$;
- τ is a factor interpretation (piece-wise multi-dimensional with definable equality) if D_τ is $(A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n)/\sim$, where $A_1 \subseteq |\mathfrak{M}|^{i_1}, \cdots, A_n \subseteq |\mathfrak{M}|^{i_n}$ are definable sets and $\sim$ is a definable equivalence relation on $A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$.

Interpretations of theories

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be countable first-order languages. Let U be an $\mathcal{L}_1$-theory and T be an $\mathcal{L}_2$-theory.

Definition (Interpretations of theories)

An interpretation of U in T is an uniformly defined family of interpretations $\langle \tau_{\mathfrak{M}} : \mathfrak{M} \models T \rangle$ where $\tau_{\mathfrak{M}}$ is an interpretation of U in $\mathfrak{M}$ for any $\mathfrak{M} \models T$.

- In this definition, "uniformly defined" means that the corresponding components of all $\tau_{\mathfrak{M}}$ should be definable sets/functions given by the same formula in all $\mathfrak{M} \models T$.
- In this work, we consider interpretations in most general sense (piece-wise multi-dimensional interpretations with definable equality).

Globaliser

Definition (Folklore)

We say a theory T is locally interpretable in a theory U (denoted by $T \trianglelefteq_{loc} U$) if any finite sub-theory T' of T is interpretable in U.

Definition (Globaliser of a c.e. theory)

A globaliser of a c.e. theory T is a c.e. theory U such that for any c.e. theory S, S is locally interpretable in T if and only if S is interpretable in U.

Proposition

Let U, T be c.e. theories. The following are equivalent:

(1) *U is a globaliser of T;*

(2) *$U \trianglelefteq_{loc} T$ and for any c.e. theory S, if $S \trianglelefteq_{loc} T$, then $S \trianglelefteq U$.*

Basic properties of globalizers

Proposition

Let T be any c.e. theory and $G(T)$ be any globalizer of T.

(1) *For any reflexive theory T, T is a globalizer of T.*

(2) *The theory $G(T)$ is a globalizer of $G(T)$.*

(3) *$T \trianglelefteq G(T)$. Thus, if T is essentially undecidable, then $G(T)$ is also essentially undecidable.*

(4) *If $T \trianglelefteq S$, then $G(T) \trianglelefteq G(S)$.*

(5) *The globaliser of T is unique in the sense that if both $G_1(T)$ and $G_2(T)$ are globalisers of T, then $G_1(T)$ and $G_2(T)$ are mutually interpretable.*

(6) *If U is a globalizer of S and S is a globalizer of T, then U is a globalizer of T.*

A Globalizer of EQ

Now, we review some known results about globalisers of c.e. theories.

Definition

The theory of pure equality is denoted by EQ, *which can be axiomatized by* $\forall x(x = x)$.

Fact (Folklore)

A theory T *is locally finitely satisfiable if and only if it is locally interpretable in* EQ.

Theorem (Visser, 2009)

For any r.e. theory T, T *is locally interpretable in* EQ *if and only if* T *is interpretable in* $\mathbf{R}$.

Corollary

The theory R *is a globalizer of* EQ.

A Globalizer of reflexive theory

Definition

A theory $U \supseteq \mathrm{Q}$ *is called reflexive, if* $U \vdash \mathrm{Con}(U')$*, for any finite subtheory* U' *of* U.

Theorem (Solomon Feferman, 1960)

For any c.e. theory T *and reflexive theory* U*, we have*

$$T \trianglelefteq_{loc} U \Rightarrow T \trianglelefteq U.$$

Corollary

For any reflexive theory U*,* U *is a globalizer of* U*.*

A Globalizer of sequential theory

Adjunctive Set Theory (AS) consists of the following axioms in the language with only one binary relation symbol $\in$.

AS1 $\exists x \forall y (y \notin x)$.

AS2 $\forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u = x \vee u = y))$.

The notion of sequential theory is an explication of the idea of theory with coding, which was introduced by Pudlák.

Definition (Sequential theory, Pudlák)

A theory is sequential if it interprets the theory AS.

Theorem (Visser, 2014)

Any sequential theory has a globalizer.

A Globalizer of Vaught theory

Definition (Vaught theory, Folklore)

(1) *Vaught set theory* (VS), *is axiomatised by* $\{vsn : n \in \omega\}$ *where*
$vsn := \forall x_0 \cdots \forall x_{n-1} \exists z \forall u(u \in z \leftrightarrow \bigvee_{i<n} u = x_i)$, *and*
$vs0 := \exists z \forall u(u \notin z)$.

(2) *A theory is a Vaught theory if it interprets* VS.

Theorem (Visser)

Any Vaught theory has a globaliser.

A globaliser for c.e. theories

Question

Does any c.e. theory has a globaliser?

The answer is positive. The following theorem is a generalization of the well known theorem that R is a globaliser of EQ.

Theorem (Pakhomov, Visser)

For any c.e. theory T, there is a globaliser $G(T)$ of T.

The globaliser $G(T)$ of T constructed in this theorem is very abstract and not an analogue of Robinson's theory R.

The motivation

- In the following, we want to construct a concrete globaliser of c.e. theories that is a natural analogue of Robinson's theory R.
- The theory $\mathcal{R}(T)$ we construct, a natural analogue of Robinson's theory R, is a weak set theory with urelements.
- The general idea is to generalize Robinson's theory R to the case with urelements.
- The theory $\mathcal{R}(T)$ corresponds to constructible sets $L(\mathfrak{M}, \omega)$ over structures $\mathfrak{M} \models T$ in the same way as R corresponds to natural numbers.
- We want to show that $\mathcal{R}(T)$ is a globaliser of T for any c.e. theory T, that generalizes the well known theorem that R is a globaliser of EQ.

Constructible hierarchy over $\mathfrak{M}$ via definability

For set theory with urelements, we refer to Barwise's book *Admissible sets and structures: An approach to definability theory*, 1975.

Definition (Constructible hierarchy over $\mathfrak{M}$ via definability)

(1) $\mathsf{L}(\mathfrak{M}, 0) = |\mathfrak{M}|$;

(2) $\mathsf{L}(\mathfrak{M}, \alpha + 1) = \mathsf{Def}(\mathsf{L}(\mathfrak{M}, \alpha)) \cup \mathsf{L}(\mathfrak{M}, \alpha)$, *where* $\mathsf{Def}(\mathsf{L}(\mathfrak{M}, \alpha))$ *is the set of all first-order definable subsets of* $\mathsf{L}(\mathfrak{M}, \alpha)$, *where in definitions we can use predicates* $\in, \mathsf{Ur}$ *(to be an urelement), and all predicates from* $\mathfrak{M}$*;*

(3) $\mathsf{L}(\mathfrak{M}, \lambda) = \bigcup_{\alpha<\lambda} \mathsf{L}(\mathfrak{M}, \alpha)$, *for limit* λ.

$\mathsf{L}(\mathfrak{M}) = \bigcup_{\alpha \in \mathsf{On}} \mathsf{L}(\mathfrak{M}, \alpha)$.

Constructible hierarchy over $\mathfrak{M}$ via Gödel's operations

Further, we will use a slightly different variant of constructible hierarchy based on Gödel's operations. Gödel operations are certain functions $\mathcal{F}_1(x, y), \cdots, \mathcal{F}_N(x, y)$ on $\mathsf{V}(\mathfrak{M})$.

Definition (Constructible hierarchy over $\mathfrak{M}$ via Gödel's operations)

(1) $\mathsf{L}(\mathfrak{M}, 0) = |\mathfrak{M}|$;
(2) $\mathsf{L}(\mathfrak{M}, \alpha + 1) = \mathsf{L}(\mathfrak{M}, \alpha) \cup \{\mathsf{L}(\mathfrak{M}, \alpha)\} \cup \{\mathcal{F}_i(x, y) : 1 \leq i \leq N$ *and* $x, y \in \mathsf{L}(\mathfrak{M}, \alpha) \cup \{\mathsf{L}(\mathfrak{M}, \alpha)\}\}$;
(3) $\mathsf{L}(\mathfrak{M}, \lambda) = \bigcup_{\alpha<\lambda} \mathsf{L}(\mathfrak{M}, \alpha)$ *for limit* λ.

This gives us the same class of constructible sets: $\mathsf{L}(\mathfrak{M}) = \bigcup_{\alpha \in \mathsf{On}} \mathsf{L}(\mathfrak{M}, \alpha)$. Further, we will use well-founded linear preorder $\leq_{\mathsf{L}}$ on $\mathsf{L}(\mathfrak{M}, \omega)$ that is

$$x \leq_{\mathsf{L}} y \overset{\text{def}}{\iff} \forall n(y \in \mathsf{L}(\mathfrak{M}, n) \rightarrow x \in \mathsf{L}(\mathfrak{M}, n)).$$

Gödel operations

We use Barwise's presentation of constructible sets in terms of finitely many binary Gödel operations $\mathcal{F}_1, \ldots, \mathcal{F}_N$ defined as follows. Recall that $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$.

Definition (Gödel operations)

(1) $\mathcal{F}_1(x, y) = \{x, y\}$;
(2) $\mathcal{F}_2(x, y) = \bigcup x$;
(3) $\mathcal{F}_3(x, y) = x - y$;
(4) $\mathcal{F}_4(x, y) = x \times y$;
(5) $\mathcal{F}_5(x, y) = \mathrm{dom}(x)$;
(6) $\mathcal{F}_6(x, y) = \mathrm{rng}(x)$;
(7) $\mathcal{F}_7(x, y) = \{\langle u, v, w \rangle : \langle u, v \rangle \in x, w \in y\}$;
(8) $\mathcal{F}_8(x, y) = \{\langle u, w, v \rangle : \langle u, v \rangle \in x, w \in y\}$;
(9) $\mathcal{F}_9(x, y) = \{z \in x : z \text{ is an urelement}\}$;
(10) $\mathcal{F}_{10}(x, y) = \{\langle v, u \rangle \in y \times x : u = v\}$;
(11) $\mathcal{F}_{11}(x, y) = \{\langle v, u \rangle \in y \times x : u \in v\}$;
(12) $\mathcal{F}_{12}$-$\mathcal{F}_N$: *For each relation symbol* $R(x_1, \cdots, x_n)$ *of* T, *we define an operation* $\mathcal{F}_R(x, y)$ *as follows:* $\mathcal{F}_R(x, y) = \{\langle p_n, \cdots, p_1, v \rangle : \langle p_n, \cdots, p_1 \rangle \in x, R(p_1, \cdots, p_n)$ *and* $v \in y\}$.

The signature of the theory $\mathcal{R}(T)$

Now, we define the theory $\mathcal{R}(T)$. WLOG, we assume that T is a c.e. theory with finite predicate-only signature.

Definition

The signature of $\mathcal{R}(T)$ consists of:

(1) *all predicates from* $\mathrm{Sgn}(T)$*;*
(2) *predicate* $x \in y$*;*
(3) *constant* Ur*;*
(4) *binary functions* $\mathcal{F}_1, \cdots, \mathcal{F}_N$ *for Gödel operations;*
(5) *unary function* $\mathcal{E}$*;*
(6) *binary predicate* $\leq_{\mathrm{L}}$*.*

Terms $\underline{\mathrm{L}_0} = \mathrm{Ur}$ and $\underline{\mathrm{L}_{n+1}} = \mathcal{E}(\underline{\mathrm{L}_n})$ denote finite levels of constructible hierarchy. The intended interpretation for $\mathcal{E}$ is

$$\mathcal{E}\colon\, b \longmapsto b \cup \{b\} \cup \{\mathcal{F}_i(x, y) \mid 1 \leq i \leq N,\; x, y \in b \cup \{b\}\}.$$

The axioms of the theory $\mathcal{R}(T)$

Definition

For each n we denote as $\underline{\mathsf{L}_n}$ *the term* $\mathcal{E}^n(\mathrm{Ur})$. *The axioms of* $\mathcal{R}(T)$ *are:*

(1) T^{Ur}, *i.e. relativizations to* Ur *of all axioms of* T*;*

(2) $x \in \mathrm{Ur} \to y \notin x$;

(3) $x \notin \mathrm{Ur} \wedge y \notin \mathrm{Ur} \wedge \forall z (z \in x \leftrightarrow z \in y) \to x = y$ *(Extensionality);*

(4) $x \in \underline{\mathsf{L}_{n+1}} \leftrightarrow x \in \underline{\mathsf{L}_n} \vee x = \underline{\mathsf{L}_n} \vee \bigvee_{1 \leq i \leq N} \exists y, z (x = \mathcal{F}_i(y, z) \wedge (y \in \underline{\mathsf{L}_n} \vee y = \underline{\mathsf{L}_n}) \wedge (z \in \underline{\mathsf{L}_n} \vee z = \underline{\mathsf{L}_n}))$, *for all n (defining axioms for* $\underline{\mathsf{L}_{n+1}}$*);*

(5) *Series of axioms* $\mathcal{F}_i$-Def_n*, for natural n and* $1 \leq i \leq N$*, stating that* $\mathcal{F}_i(x, y)$ *works on* $x, y \in \underline{\mathsf{L}_n}$ *as in Definition 17;*

(6) $x \in \underline{\mathsf{L}_n} \wedge y \leq_{\mathrm{L}} x \to \bigvee_{m \leq n} y \in \underline{\mathsf{L}_m}$, *for all natural n;*

(7) $x \in \underline{\mathsf{L}_n} \to y \leq_{\mathrm{L}} x \vee x \leq_{\mathrm{L}} y$.

The theory $\mathcal{R}(T)$ we construct is a natural analogue of R.

The main theorem

Theorem

For any c.e. theory T, the theory $\mathcal{R}(T)$ is a globaliser of T.

It suffices to show the following two propositions:

(1) The theory $\mathcal{R}(T)$ is locally interpretable in T;

(2) Any c.e. theory locally interpretable in T is interpretable in $\mathcal{R}(T)$.

Theorem

For any c.e. theory T, $\mathcal{R}(T) \trianglelefteq_{loc} T$.

Fact

For any finite fragment U of $\mathcal{R}(T)$, there exists n such that for any $\mathfrak{M} \models T$, we have $\mathsf{L}(\mathfrak{M}, n) \models U$.

- To show this, we use piecewise multi-dimensional interpretations with definable equality.
- Namely, the domain of interpretation is in the form $(A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n)/\sim$, where $A_i (1 \leq i \leq n)$ are definable sets (possibly of different dimensions) and $\sim$ is a definable equivalence relation on $A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$.
- In order to show that $\mathcal{R}(T)$ is locally interpretable in T, we show that for any n there is an interpretation τ_n in T such that $\tau_n(\mathfrak{M})$ is $\mathsf{L}(\mathfrak{M}, n)$, for any $\mathfrak{M} \models T$.
- We construct τ_n by induction on n. In the step of induction, we modify τ_n by adding representatives for all sets that should be present in $\mathsf{L}(\mathfrak{M}, n+1)$ and next putting factorization on top of this in order to recover extensionality.

Theorem

For any c.e. theory U, if $U \trianglelefteq_{loc} T$, then $U \trianglelefteq R(T)$.

We give a sketch of the idea of the proof.

- Suppose U is a c.e. theory locally interpretable in T. We want to show that U is interpretable in $\mathcal{R}(T)$.
- Note that for $\mathfrak{M} \models T$ there are uniformly Σ_1-definable functions $\mathrm{Mod}_{\mathfrak{M}}(x)$ in $\mathrm{L}(\mathfrak{M}, \omega)$ that map Von Neumann natural n to a model $A \in \mathrm{L}(\mathfrak{M}, \omega)$ such that A satisfies the first n axioms of U.
- Note that this could be defined by a single Σ_1-formula Mod. In $\mathcal{R}(T)$ the definition $\mathrm{Mod}(x)$ gives a partial function that is well-defined and works as intended for standard Von Neumann naturals n.
- Let J_0 be the $\mathcal{R}(T)$ definable class of naturals x such that for any $y \leq x$ the value $\mathrm{Mod}(y)$ is defined and is a model of the first y axioms of U.
- Notice that any definable non-standard element $a \in J_0$ would give us an interpretation of U as $\mathrm{Mod}(a)$.

Now, we employ a standard cut-shortening technique. We get a sequence of downward-closed classes of naturals $J_0, J_1, \cdots, J_n$ such that $\mathcal{R}(T)$ proves that either one of J_i's has the greatest element or $(\mathsf{Q} + \mathsf{Con}(U))^{J_n}$. We interpret U in $\mathcal{R}(T)$ by case consideration:

(1) $\mathsf{Mod}(\max(J_0))$ if there exists $\max(J_0)$;
(2) $\mathsf{Mod}(\max(J_1))$ if neither of above and there exists $\max(J_1)$;
(3) $\cdots\cdots$
(4) $\mathsf{Mod}(\max(J_n))$ if neither of above and there exists $\max(J_n)$;
(5) Interpretation via completeness theorem on a definable cut of J_n, if none of $J_0, \cdots, J_n$ has a greatest element.

Theorem (Interpretation existence lemma, Folklore, many authors)

For any c.e. theory U, U is interpretable in $\mathsf{Q} + \mathsf{Con}(U)$.

Some corollaries

Definition

We say a theory is a globaliser if it is a globaliser of some theory.

Proposition

Let T, S be any c.e. theory.

(1) *If T is reflexive, then $\mathcal{R}(T)$ is mutually interpretable with T.*

(2) *The following statements are equivalent:*

  (1) *T is a globaliser;*

  (2) *$\mathcal{R}(T)$ is mutually interpretable with T;*

  (3) *For any c.e. theory S, if $S \trianglelefteq_{loc} T$, then $S \trianglelefteq T$;*

  (4) *T is a globaliser of T.*

(3) *The following statements are equivalent:*

  (1) *T is not a globaliser;*

  (2) *$T \lhd \mathcal{R}(T)$;*

  (3) *There exists a c.e. theory S such that $S \trianglelefteq_{loc} T$, but $S \ntrianglelefteq T$;*

(4) *$\mathcal{R}^n(\mathrm{R})$ is mutually interpretable with R for any $n \geq 1$.*

Reference list

- Jon Barwise, *Admissible sets and structures: An approach to definability theory*, 1975.
- Yong Cheng; Finding the limit of incompleteness I, Accepted and to appear in *The Bulletin of Symbolic Logic*, arXiv:1902.06658v2, 2020.
- Yong Cheng and Fedor Pakhomov; A R-like Globaliser for c.e. theories, reprint, 2020.
- Petr Hájek and Pavel Pudlák, *Metamathematics of First-Order Arithmetic*, Berlin: Springer, 1993.
- Pavel Pudlák, Cuts, consistency statements and interpretations,*J. Symb. Log*, 50(2), 423-441, 1985.
- Alfred Tarski, Andrzej Mostowski and Raphael M. Robinson, *Undecidabe theories*, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1953.
- Albert Visser, On **Q**, *Soft Comput*, DOI 10.1007/s00500-016-2341-5, 2016.
- Albert Visser, The interpretation existence lemma. In *Feferman on Foundations*, Outstanding Contributions to Logic13, pages 101-144. Springer, 2017.

Thanks for your attention!