Incompleteness for higher order arithmetic: A specific example from set theory

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Gödel's incompleteness theorem

Two goals of Hilbert's program:

Completeness A proof that all true mathematical statements can be proved in the formalism of mathematics.

Consistency A proof that no contradiction can be obtained in the formalism of mathematics using only "finitistic" reasoning about finite mathematical objects.

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Theorem (Gödel-Rosser)

(1) *Gödel-Rosser first incompleteness theorem* (**G1**)*: If T is a recursively axiomatized consistent extension of* **PA**, *then T is not complete.*

(2) *Gödel's second incompleteness theorem* (**G2**)*: If T is a recursively axiomatized consistent extension of* **PA**, *then the consistency of T is not provable in T.*

Mathematical example of **G1** for **PA**

- Gödel constructs a true sentence about arithmetic which is not provable in **PA**.
- From the viewpoint of classic mathematics, Gödel's sentence is artificial and has no real mathematical content.

Question

Could we find a sentence about arithmetic with interesting mathematical contents which is independent of **PA**?

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Theorem (Paris-Harrington)

If **PA** *is consistent, then there exists a sentence ϕ of combinatorial contents such that $\mathfrak{N} \models \phi$, but ϕ is independent of* **PA** *where $\mathfrak{N} = (\mathbb{N}, +, \cdot)$.*

Definition of higher order arithmetic

Definition

(1) $Z_2 = ZFC^- +$ *Every set is countable.*[a]

(2) $Z_3 = ZFC^- + \mathcal{P}(\omega)$ *exists* $+$ *Every set is of cardinality* $\leq \beth_1$.

(3) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ *exists* $+$ *Every set is of cardinality* $\leq \beth_2$.

[a] ZFC^- denotes ZFC with the Power Set Axiom deleted and Collection instead of Replacement.

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Corollary

If Z_2 *is consistent, then there is a true sentence about analysis which is not provable in* Z_2.

Relativized Hilbert's program to Z_2

Fact

Many classic mathematical theorems about analysis which are expressible in Z_2 are provable in Z_2.

Question

Relativized Hilbert's program to Z_2 *Is Z_2 complete for classic mathematical theorems expressible in Z_2?*

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Motivation In this talk, I give a counterexample from set theory for this question which is expressible in Z_2 but not provable in Z_2.

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Boldface Martin-Harrington theorem $Det(\boldsymbol{\Sigma}^1_1)$ if and only if for any real $x, x^\sharp$ exists.

Definition

We let Harrington's Principle, HP *for short, denote the following statement: there is a real x such that if α is a countable x-admissible ordinal, then α is an L-cardinal.*

Harrington's proof of "$Det(\Sigma_1^1)$ implies $0^\sharp$ exists" in ZF is done in two steps:

First Step $Det(\Sigma_1^1)$ implies HP;

Second Step HP implies $0^\sharp$ exists.

Analysis of Lightface Harrington's theorem

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First Step $Det(\Sigma^1_1)$ implies HP;

Second Step HP implies $0^\sharp$ exists.

So in ZF we have:

$$Det(\Sigma^1_1) \Leftrightarrow \text{HP} \Leftrightarrow 0^\sharp \text{ exists.}$$

Fact

(Z_2) $Det(\Sigma^1_1)$ *implies* HP.

$Z_2 + Det(\Sigma^1_1)$ implies HP

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Harrington, Frideman Proof via Steel forcing

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Sami Proof via effective descriptive set theory, totally forcing-free

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Harrington, Frideman Proof via Steel forcing

Sami Proof via effective descriptive set theory, totally forcing-free

Woodin Proof via Barwise-compactness theorem and basic Cohen forcing $Col(\omega, \alpha)$

Z_2 + HP does not imply $0^\sharp$

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- If yes, then lightface Harrington's theorem is provable in Z_2.
- If no, then is "HP implies $0^\sharp$ exists" provable in Z_3 or Z_4?

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Fact

$Z_2 + 0^\sharp$ *exists implies* HP.

Theorem

(Set forcing) Assuming ω_1 is a limit cardinal in L, we can force a set model of Z_2 + HP.

Corollary

Z_2 + HP *does not imply* $0^\sharp$ *exists.*

The notion of remarkable cardinal

Theorem

(Magidor) A cardinal κ is supercompact if and only if for every regular cardinal $\lambda > \kappa$ there is a regular cardinal $\overline{\lambda} < \kappa$ and an elementary embedding $j : H_{\overline{\lambda}} \to H_{\lambda}$ with $j(crit(j)) = \kappa$.

We can view a remarkable cardinal as a type of generic supercompact cardinal using Magidor's theorem.

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Definition

(Schindler) A cardinal κ is remarkable if for every regular cardinal $\lambda > \kappa$, there is a regular cardinal $\overline{\lambda} < \kappa$ such that in $V^{Col(\omega,<\kappa)}$ there is an elementary embedding $j : H^V_{\overline{\lambda}} \to H^V_\lambda$ with $j(crit(j)) = \kappa$.

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Theorem

(Set forcing) Assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, we can force a model of Z_3 + HP.

Corollary

Z_3 + HP *does not imply* $0^\sharp$ *exists.*

Strong reflecting property

Definition

Let $\gamma \geq \omega_1$ be an L-cardinal. γ has the strong reflecting property for L-cardinals, denoted $\mathrm{SRP}^L(\gamma)$, if and only if for some regular cardinal $\kappa > \gamma$, if $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$, then $\overline{\gamma}$ is an L-cardinal. If $\gamma < \omega_1$, we say that γ has the strong reflecting property iff $\gamma = \overline{\gamma}$.

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The first step Force a club in ω_2 of L-cardinals with the strong reflecting property via set forcing.

The second step Force a set model of $\mathrm{Z}_3 + \mathrm{HP}$ via set forcing without reshaping.

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- The proof uses that $\mathrm{SRP}^L(\omega_2)$. Only knowing that $\mathrm{SRP}^L(\gamma)$ for some $\gamma \in [\omega_1, \omega_2)$ is not enough for the proof.

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- The hypothesis "there exists a remarkable cardinal with a weakly inaccessible cardinal above it" is only used in the first step.
- The proof is done via set forcing without use of reshaping.
- Only assuming one remarkable cardinal is not enough for our proof.
- The proof uses that $\mathrm{SRP}^L(\omega_2)$. Only knowing that $\mathrm{SRP}^L(\gamma)$ for some $\gamma \in [\omega_1, \omega_2)$ is not enough for the proof.
- For my proof of forcing a set model of $Z_3 + \mathrm{HP}$, the assumption that there exists a remarkable cardinal with a weakly inaccessible cardinal above it is optimal.

Characterization of strong reflecting property for ω_n

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- (i) $\mathsf{SRP}^L(\omega_2)$;
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(3) *The following are equivalent:*

- (i) $\mathsf{SRP}^L(\gamma)$ *holds for some* L*-cardinal* $\gamma > \omega_2$.
- (ii) $0^\sharp$ *exists.*
- (iii) $\mathsf{SRP}^L(\gamma)$ *holds for every* L*-cardinal* $\gamma \geq \omega_1$.

Large cardinal strength of "$Z_3 + \mathrm{HP}$"

Question

What is the large cardinal strength of "$Z_2 + \mathrm{HP}$" and "$Z_3 + \mathrm{HP}$"?

Theorem (joint with Schindler)

Class forcing $Z_2 + \mathrm{HP}$ *is equiconsistent with* ZFC.

Class forcing *The following two theories are equiconsistent:*

(1) $Z_3 + \mathrm{HP}$.

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Large cardinal strength of “$Z_3 +$ HP”

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Class forcing *The following two theories are equiconsistent:*

(1) $Z_3 +$ HP.

(2) ZFC $+$ *there exists a remarkable cardinal.*

The Z_3 proof is done in two steps:

(1) Z_3 + HP implies $L \models$ ZFC $+ \omega_1^V$ is remarkable.

(2) If “ZFC $+$ there exists a remarkable cardinal” is consistent, then “$Z_3 +$ HP” is consistent.

Two ways to force a model of $Z_3 + HP$

Set forcing: Assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, without reshaping

Class forcing: Assuming there exists a remarkable cardinal, with reshaping

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1. *Is it true that if we can force a model of proposition ϕ by class forcing assuming some large cardinal axioms, then we could force a model of ϕ by set forcing assuming the same large cardinal axioms?*

Two ways to force a model of $Z_3 + \mathrm{HP}$

Set forcing: Assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, without reshaping

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Question

1. *Is it true that if we can force a model of proposition ϕ by class forcing assuming some large cardinal axioms, then we could force a model of ϕ by set forcing assuming the same large cardinal axioms?*
2. *Could we force a model of $Z_3 + \mathrm{HP}$ via set forcing only assuming the existence of one remarkable cardinal?*

Z_4 + HP implies $0^\sharp$

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Is "HP implies $0^\sharp$ exists" provable in Z_4?

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Two definitions of $0^\sharp$ in Z_2:

(1) $0^\sharp$ *is the unique well founded remarkable cofinal* EM *set;*

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Theorem

(Z_4) *The following are equivalent:*

(1) HP.

(2) L_{ω_2} *has an uncountable set of indiscernibles.*

(3) $0^\sharp$ *exists.*

Boldface Harrington's theorem

Fact

Both Lightface and Boldface Martin's theorem are provable in Z_2.

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Theorem

Boldface Harrington's theorem is provable in Z_2.

Definition

For any real x, *let* $\mathrm{HP}(x)$ *denote the statement: there is a real* y *such that if* α *is a countable* y*-admissible ordinal, then* α *is an* $L[x]$*-cardinal.*

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The proof is done in two steps:

1. $Z_2 + Det(\boldsymbol{\Sigma}^1_1)$ implies for any real x, $\mathrm{HP}(x)$ holds;
2. $Z_2 + \forall x \in \omega^\omega(\mathrm{HP}(x))$ implies that for any real x, $x^\sharp$ exists.

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- "HP implies $0^\sharp$ exists" is not provable in Z_3.
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- Z_4 is the minimal system in higher order arithmetic to show that HP implies $0^\sharp$ exists.

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- $Z_2 +$ HP is equiconsistent with ZFC.

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- In Z_4, HP is equivalent to $0^\sharp$ exists.
- Z_4 is the minimal system in higher order arithmetic to show that HP implies $0^\sharp$ exists.
- $Z_2 +$ HP is equiconsistent with ZFC.
- $Z_3 +$ HP is equiconsistent with ZFC $+$ there exists a remarkable cardinal.

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- I find an interesting classic mathematical theorem expressible in Z_2 but not provable in Z_2: "HP implies $0^\sharp$ exists".
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- In Z_4, HP is equivalent to $0^\sharp$ exists.
- Z_4 is the minimal system in higher order arithmetic to show that HP implies $0^\sharp$ exists.
- $Z_2 +$ HP is equiconsistent with ZFC.
- $Z_3 +$ HP is equiconsistent with ZFC $+$ there exists a remarkable cardinal.
- Lightface Martin's theorem, Boldface Martin's theorem and Boldface Harrington's theorem are all provable in Z_2.

Reference list

- Yong Cheng, Forcing a set model of $Z_3 +$ Harrington's Principle, *Mathematical Logic Quarterly* 61, No. 4-5, 274-287, 2015.
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Thanks for your attention!