Two questions about incompleteness

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Modern logic and Philosophy

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- Implication, Consistency, Paradox, Contradiction
- Absoluteness, Knowability, Necessity, Vagueness, etc.

Part One: The current state

Outline

The current state of research:

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(1) Property of provability and truth

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(2) Generalization of Incompleteness theorem to arithmetical definable theory

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(4) Proving Incompleteness via logical paradox

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(5) The intensionality of **G2** for **PA**

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(5) The intensionality of **G2** for **PA**

(6) Incompleteness and provability logic

Gödel's incompleteness theorem

Two goals of Hilbert's program:

Completeness A proof that all true mathematical statements can be proved in the formalism of mathematics.

Consistency A proof that no contradiction can be obtained in the formalism of mathematics using only "finitistic" reasoning about finite mathematical objects.

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Theorem (Gödel-Rosser)

(1) *Gödel-Rosser first incompleteness theorem* (**G1**)*: If T is a recursively axiomatized consistent extension of* **PA**, *then T is not complete.*

(2) *Gödel's second incompleteness theorem* (**G2**)*: If T is a recursively axiomatized consistent extension of* **PA**, *then the consistency of T is not provable in T.*

Provability and Truth

Definition

1. **Prof**$=\{\ulcorner\phi\urcorner : \phi$ *is sentence and* **PA** $\vdash \phi\}$.
2. **Truth**$=\{\ulcorner\phi\urcorner : \phi$ *is sentence and* $\mathfrak{N} \models \phi\}$ *where* $\mathfrak{N} = (\mathbb{N}, +, \cdot)$.

Theorem (Tarski's theorem on undefinability of truth)

Truth *is not definable in* $\mathfrak{N}$.

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Truth	**Prof**
not definable in $\mathfrak{N}$	definable in $\mathfrak{N}$
not arithmetic	recursive eumerable
not recursive	not recursive
not representable in **PA**	not representable in **PA**
productive	not productive

Solovay's arithmetical completeness theorem

Definition

An arithmetic interpretation is a function that assigns to each formula of modal logic a sentence of the language of arithmetic.

Theorem (Solovay)

Arithmetical completeness theorem for **GL** *For any modal formula* ϕ, $\mathbf{GL} \vdash \phi$ *iff for every arithmetic interpretation* f, $\mathbf{PA} \vdash \phi^f$.

Arithmetical completeness theorem for **GLS** *For any modal formula* ϕ, $\mathbf{GLS} \vdash \phi$ *iff for every arithmetic interpretation* f, $\mathfrak{N} \models \phi^f$.

Definition

(1) *We say T is Σ_n-definable iff there is a Σ_n formula $\alpha(x)$ such that $\{n \in \omega : \mathfrak{N} \models \alpha(\overline{n})\} = \{\ulcorner \phi \urcorner : \phi \in T\}$.*

(2) *We say T is Σ_n-sound if and only if for all Σ_n sentences ϕ, if $T \vdash \phi$, then $\mathfrak{N} \models \phi$.*

- Gödel's incompleteness theorem hold for Σ_1-definable theories containing **PA**.
- We generalize Gödel's incompleteness theorem for arithmetically definable theories.

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Theorem (Kikuchi, Kurahashi, 2017)

(1) *Every Σ_{n+1}-definable Σ_n-sound theory is incomplete.*

(2) *Every consistent theory having Π_{n+1} set of theorems has a true but unprovable Π_n sentence.*

(3) *Any Σ_{n+1}-definable Σ_n-sound theory can not prove its own Σ_n-soundness.*

Different proofs of incompleteness theorem

- Constructive proof: directly construct the independence sentence
- Proof via diagonalization lemma
- Proof via logical paradox
- Proof via recursion theory
- Proof via model theory

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Question

Could we give a self-reference-free proof of Gödel's incompleteness theorem?

Incompleteness theorem and logical paradox

- Incompleteness is closely related to paradox.
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Different proofs of incompleteness theorem via paradox:

Gödel	Liar Paradox
Boolos	Berry's paradox
Kurahashi	Yablo's Paradox
Kritchman	Unexpected Examination Paradox
Cieśliński	Grelling's paradox

Numeration and provability predicate

Definition

Let T be any recursively axiomatized consistent extension of **PA** *and $\alpha(x)$ be a formula in the same language.*

1. *$\alpha(x)$ is a numeration of T if for any n, $\mathbf{PA} \vdash \alpha(\overline{n})$ iff n is the Gödel number of some axiom of T.*
2. *Let $\alpha(x)$ be a numeration of T. Define the formula $\mathbf{Prf}_\alpha(x, y)$ saying "y is the Gödel number of a proof of the formula with Gödel number x from the set of all sentences satisfying $\alpha(x)$".*
3. *Define the provability predicate $\mathbf{Pr}_\alpha(x)$ of $\alpha(x)$ as $\mathbf{Pr}_\alpha(x) \triangleq \exists y \mathbf{Prf}_\alpha(x, y)$ and consistency statement $\mathbf{Con}_\alpha$ as $\triangleq \neg\mathbf{Pr}_\alpha(\perp)$.*

Drivability Conditions and **G2**

Let T be a recursively axiomatized consistent extension of **PA** and $\alpha(x)$ be a numeration of T. Then $\mathbf{Pr}_\alpha(x)$ satisfies the following properties:

- **D1** If $T \vdash \varphi$, then $\mathbf{PA} \vdash \mathbf{Pr}_\alpha(\overline{\ulcorner \varphi \urcorner})$;
- **D2** If φ is Σ_1 sentence, then $\mathbf{PA} \vdash \varphi \to \mathbf{Pr}_\alpha(\overline{\ulcorner \varphi \urcorner})$;
- **D3** $\mathbf{PA} \vdash \mathbf{Pr}_\alpha(\overline{\ulcorner \varphi \urcorner}) \to (\mathbf{Pr}_\alpha(\overline{\ulcorner \varphi \to \psi \urcorner}) \to \mathbf{Pr}_\alpha(\overline{\ulcorner \psi \urcorner}))$.

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Theorem (G2, Gödel)

Let T be any recursively axiomatized consistent extension of **PA**. *If $\alpha(x)$ is any Σ_1 numeration of T, then $T \nvdash \mathbf{Con}_\alpha$.*

The intensionality of **G2** for **PA**

Theorem (Feferman)

Let T be any recursively axiomatized consistent extension of **PA** *and $\tau(x)$ be any formula numerating T in T. Define $\tau^*(x) \triangleq \tau(x) \wedge \mathbf{Con}_{\tau|x}$. Then $\tau^*(x)$ numerates T in T and $T \vdash \mathbf{Con}_{\tau^*}$.*

Corollary

There exists a Π_1 numeration $\pi(x)$ of **PA** *such that* **G2** *fails:* $\mathbf{PA} \vdash \mathbf{Con}_{\pi}$.

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Corollary

There exists a Π_1 numeration $\pi(x)$ of **PA** *such that* **G2** *fails:* **PA** $\vdash \mathbf{Con}_{\pi}$.

- Whether **G2** holds for **PA** depends on the numeration of **PA**.
- **D1**-**D3** are the sufficient condition but not the necessary condition to show that **G2** holds for **PA**.

Theorem (Kurahashi, 2017)

There exists a Σ_2 numeration $\alpha(x)$ of **PA** *such that* **D1**-**D3** *does not hold for $\mathbf{Pr}_{\alpha}(x)$ but $\mathbf{G_2}$ holds for* **PA**.

Let T be any recursively axiomatized consistent extension of **PA** and $\alpha(x)$ be a numeration of T. The provability logic $\mathbf{PL}_\alpha(T)$ is the set of all modal principles which are verifiable in T when the modal operator $\Box$ is interpreted as $\mathbf{Pr}_\alpha(x)$.

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Theorem (Solovay's arithmetical completeness theorem)

Let T be any recursively axiomatized consistent extension of **PA**. *If T is Σ_1-sound, then for any Σ_1 numeration $\alpha(x)$ of T, the provability logic $\mathbf{PL}_\alpha(T)$ is precisely* **GL**.

Classification of provability logic under numeration

Let T be any recursively axiomatized consistent extension of **PA**. For $n \geq 1$, the provability logic $\mathbf{PL}_\tau(T)$ of a Σ_n numeration $\tau(x)$ of T is a normal modal logic.

Question

Which normal modal logic is a provability logic $\mathbf{PL}_\tau(T)$ of some Σ_n numeration $\tau(x)$ of T?

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Question

Which normal modal logic is a provability logic $\mathbf{PL}_\tau(T)$ *of some* Σ_n *numeration* $\tau(x)$ *of* T*?*

Theorem (Kurahashi, 2018)

(1) *For any recursively axiomatized consistent extension* T *of* **PA**, *there exists a* Σ_2 *numeration* $\alpha(x)$ *of* T *such that the provability logic* $\mathbf{PL}_\alpha(T)$ *is* **K**.

(2) *For each* $n \geq 2$, *there exists a* Σ_2 *numeration* $\tau(x)$ *of* T *such that the provability logic* $\mathbf{PL}_\tau(T)$ *coincides with modal logic* $\mathbf{K} + \Box(\Box^n p \to p) \to \Box p$.

Part Two: Understanding incompleteness

Research Motivation

Motivation Understanding incompleteness: Exploring the relationship between incompleteness, self-reference, provability logic, logical paradox and formal theory of truth

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1. Incompleteness for high order arithmetic

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In this talk, I focus on the following two questions about incompleteness:

1. Incompleteness for high order arithmetic
2. The limit of Incompleteness for subsystems of **PA**

Mathematical examples of **G1** for **PA**

Gödel's proof of **G1** uses meta-mathematics and the independent sentence Gödel constructed (Gödel's sentence) is of meta-mathematical nature and has no real mathematical content.

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Could we find a sentence about arithmetic with interesting mathematical contents which is independent of **PA***?*

Theorem (Paris-Harrington)

If **PA** *is consistent, then there exists a sentence ϕ of combinatorial contents such that $\mathfrak{N} \models \phi$, but ϕ is independent of* **PA***.*

Incompleteness for high order arithmetic

Definition

Definition of higher order arithmetic:

(1) $Z_2 = ZFC^- +$ *Every set is countable.*[a]

(2) $Z_3 = ZFC^- + \mathcal{P}(\omega)$ *exists* $+$ *Every set is of cardinality* $\leq \beth_1$.

(3) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ *exists* $+$ *Every set is of cardinality* $\leq \beth_2$.

[a] ZFC^- denotes ZFC with the Power Set Axiom deleted and Collection instead of Replacement.

Corollary

If Z_2 is consistent, then there is a true sentence about analysis which is not provable in Z_2.

Fact

Many classic mathematical theorems about analysis which are expressible in Z_2 *are provable in* Z_2.

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Motivation Finding a counterexample for this question which is expressible in Z_2 but not provable in Z_2.

Harrington's theorem $Det(\Sigma^1_1)$ implies $0^\sharp$ exists.

Definition

We let Harrington's Principle, HP *for short, denote the following statement:* $\exists x \in 2^\omega \forall \alpha(\alpha$ *is countable x-admissible* $\rightarrow \alpha$ *is an L-cardinal).*

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First Step $Det(\Sigma^1_1)$ implies HP;

Second Step HP implies $0^\sharp$ exists.

In ZF we have

$$Det(\Sigma^1_1) \Leftrightarrow \text{HP} \Leftrightarrow 0^\sharp \text{ exists.}$$

The counterexample

- The first step "$Det(\Sigma^1_1)$ implies HP" is provable in Z_2.

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Question

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The counterexample I find is the sentence: "HP implies $0^\sharp$ exists":

Theorem

(1) *"HP implies $0^\sharp$ exists" is not provable in Z_2.*

(2) *"HP implies $0^\sharp$ exists" is not provable in Z_3.*

(3) *"HP implies $0^\sharp$ exists" is provable in Z_4.*

So Z_4 is the minimal system in higher order arithmetic to show that "HP implies $0^\sharp$ exists".

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- In Z_4, HP is equivalent to $0^\sharp$ exists.
- Z_4 is the minimal system in higher order arithmetic to show that HP implies $0^\sharp$ exists.

Theorem (joint with Ralf Schindler)

1. $Z_2 +$ HP *is equiconsistent with* ZFC*.*
2. $Z_3 +$ HP *is equiconsistent with* ZFC $+$ *there exists a remarkable cardinal.*

The limit of Incompleteness for subsystems of **PA**

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- An interpretation of a theory T in a theory S is a mapping from formulas of T to formulas of S that maps all axioms of T to sentences provable in S.
- Let $Int(S)$ denote the degree of interpretation of theory S. $Int(T) < Int(S)$ means that T is interpretable in S but S is not interpretable in T. $Int(T) = Int(S)$ means that T and S are mutually interpretable.
- Interpretability can be accepted as a measure of strength of first order theory.

Definition

Let T be a recursively axiomatizable consistent theory.

1. *T is essentially undecidable iff any recursively axiomatizable consistent extension of T is undecidable.*
2. *T is essentially incomplete iff any recursively axiomatizable consistent extension of T is imcomplete.*
3. **G1** *holds for T iff for any recursively axiomatizable consistent theory S, if T is interpretable in S, then S is undecidable.*

Proposition

Let T be a recursively axiomatizable consistent theory. The followings are equivalent:

1. **G1** *holds for T.*
2. *T is essentially undecidable.*
3. *T is essentially incomplete.*

Robinson's $\mathbf{Q}$

Question

Could we find a theory S with minimal degree of interpretation such that **G1** *holds for S?*

Definition

Let Robinson's $\mathbf{Q}$ *be the system consisting of the following sentences:*

1. $\forall x \forall y (\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$;
2. $\forall x (\mathbf{S}x \neq \mathbf{0})$;
3. $\forall x (x \neq \mathbf{0} \rightarrow \exists y x = \mathbf{S}y)$;
4. $\forall x \forall y (x + \mathbf{0} = x)$;
5. $\forall x \forall y (x + \mathbf{S}y = \mathbf{S}(x + y))$;
6. $\forall x (x \cdot \mathbf{0} = \mathbf{0})$;
7. $\forall x \forall y (x \cdot \mathbf{S}y = x \cdot y + x)$.

System $\mathbf{R}$

We work on $L(\overline{0}, \cdots, \overline{n} \cdots, +, \cdot, \leq)$ with infinitely many constants as names for natural numbers and with $\leq$ as primitive symbol.

Definition

Let $\mathbf{R}$ *be the system consisting of schemes* $Ax1$-$Ax5$ *where* $m, n \in \mathbb{N}$.

$Ax1$ $\overline{m} + \overline{n} = \overline{m+n}$;

$Ax2$ $\overline{m} \neq \overline{n}$ *if* $m \neq n$;

$Ax3$ $\overline{m} \cdot \overline{n} = \overline{m \cdot n}$;

$Ax4$ $\forall x(x \leq \overline{n} \rightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$;

$Ax5$ $\forall x(x \leq \overline{n} \vee \overline{n} \leq x)$.

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$Ax5$ $\forall x(x \leq \overline{n} \vee \overline{n} \leq x)$.

Theorem

(Visser Albert) Suppose T is an R.E. theory. Then T is locally finite iff T is interpretable in **R**.

Properties of **Q** and **R**

1. **R** $\subseteq$ **Q** $\subseteq$ **PA**; **Q** is finitely axiomatizable but **R** is not.
2. **Q** is minimal essentially undecidable; **R** is not minimal essentially undecidable.
3. $Int(\mathbf{R}) < Int(\mathbf{Q})$ since **Q** is not interpretable in **R**.

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G1 *holds for* **R**.

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Theorem (Folklore)

G1 *holds for* $\mathbf{R}$.

Question

Could we find a theory S such that **G1** *holds for S and $Int(S) < Int(\mathbf{R})$?*

System $\overline{\mathbf{R}}$

Definition

Let $\overline{\mathbf{R}}$ be the system consisting of schemes $\Omega_2, \Omega_3, \Omega_4'$ where $m, n \in \mathbb{N}$.

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- $Ax3$ $\overline{m} \cdot \overline{n} = \overline{m \cdot n}$;
- Ax' $\forall x(x \leq \overline{n} \leftrightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$.

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Let $\overline{\mathbf{R}}$ be the system consisting of schemes $\Omega_2, \Omega_3, \Omega_4'$ where $m, n \in \mathbb{N}$.

$$Ax2 \quad \overline{m} \neq \overline{n} \text{ if } m \neq n;$$

$$Ax3 \quad \overline{m} \cdot \overline{n} = \overline{m \cdot n};$$

$$Ax' \quad \forall x(x \leq \overline{n} \leftrightarrow x = \overline{0} \vee \cdots \vee x = \overline{n}).$$

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Theorem

(1) **G1** *holds for* $\overline{\mathbf{R}}$.

(2) $\mathbf{R}$ *is interpretable in* $\overline{\mathbf{R}}$*, and hence* $Int(\overline{\mathbf{R}}) = Int(\mathbf{R})$.

Definition

$\langle S, T\rangle$ is a recursively inseparable pair if $S, T \subseteq \mathbb{N}$ both are recursively enumerable and there is no recursive set $X \subseteq \mathbb{N}$ such that $S \subseteq X$ and $X \cap T = \emptyset$.

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For any recursively inseparable pair $\langle S, T\rangle$, there exists theory $U_{\langle S,T\rangle}$ such that **G1** *holds for $U_{\langle S,T\rangle}$ and $Int(U_{\langle S,T\rangle}) < Int(\mathbf{R})$.*

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Definition

Let $\langle S, T \rangle$ be a recursively inseparable pair. Let L be the finite language $\{\mathbf{0}, \mathbf{S}, \mathbf{P}\}$. Consider the following theory $U_{\langle S,T \rangle}$:

- $\overline{m} \neq \overline{n}$ *if* $m \neq n$;
- $\mathbf{P}(\overline{n})$ *if* $n \in S$;
- $\neg \mathbf{P}(\overline{n})$ *if* $n \in T$.

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G1 *holds for* $U_{\langle S,T\rangle}$ *and* $Int(U_{\langle S,T\rangle}) < Int(\mathbf{R})$.

Model completion of the empty theory

Definition

1. *A consistent theory T is said to be model complete if for all models $\mathfrak{A}$, $\mathfrak{B}$ of T, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.*
2. *A theory T^* is a model companion of T if T^* is a cotheory of T and T^* is model complete.*
3. *A theory T^* is a model completion of T if T^* is a model companion of T and for every model $\mathfrak{A}$ of T with diagram $\Delta_{\mathfrak{A}}$, $T^* \cup \Delta_{\mathfrak{A}}$ is complete.*

For any language L, let EC_L be the model completion of the empty L-theory. Then

Fact

(1) *EC_L has elimination of quantifiers.*

(2) *Models of EC_L are exactly the existentially closed L-structures; in particular, every L-structure embeds in a model of EC_L.*

Proof of the main theorem

Theorem (Emil JeŘábek)

For any language L and formula $\phi(\overline{z}, \overline{x}, \overline{y})$ with $lh(\overline{x}) = lh(\overline{y})$, there is a constant n with the following property. Let $M \models EC_L$ and $\overline{u} \in M$ be such that $M \models \overline{x}_0, \cdots, \overline{x}_{n-1} \bigwedge_{i<j<n} \phi(\overline{u}, \overline{x}_i, \overline{x}_j)$. Then for every $m \in \omega$ and an asymmetric relation R on $\{0, \cdots, m-1\}$, $M \models \overline{x}_0, \cdots, \overline{x}_{m-1} \bigwedge_{\langle s,t\rangle \in R} \phi(\overline{u}, \overline{x}_s, \overline{x}_t)$.

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Consider the following theory S in the language $\langle \in \rangle$ axiomatized by the sentences $\exists z, x_0, ..., x_n(\bigwedge_{i<j<n} x_i \neq x_j \wedge \forall y(y \in z \leftrightarrow \bigvee_{i<n} y = x_i))$ for all $n \in \omega$.

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Corollary

S is not weakly interpretable in EC_L.

Proof of the main theorem: continued

In the following, based on Emil's work I show that **R** is not weakly interpretable in EC_L.

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- Note that S is interpretable in **R**.
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Lemma

If **R** *is interpretable in* $U_{\langle S,T\rangle}$*, then* **R** *is weakly interpretable in* EC_L *for some language* L*.*

Proof of the main theorem: continued

In the following, based on Emil's work I show that $\mathbf{R}$ is not weakly interpretable in EC_L.

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If $\mathbf{R}$ is interpretable in $U_{\langle S,T\rangle}$, then $\mathbf{R}$ is weakly interpretable in EC_L for some language L.

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$\mathbf{R}$ is not interpretable in $U_{\langle S,T\rangle}$.

Questions for future research

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1. *Is* $(\mathbf{D}, <)$ *well founded?*
2. *Are any two elements of* $\mathbf{D}$ *comparable?*

Conjecture

$(\mathbf{D}, <)$ *is not well founded and has incomparable elements.*

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Thanks for your attention!